Denoising of Manifold Valued Data by Relaxed Regularizations

DWCAA 24 - Canazei, Italy

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- **Aim 1):** nicely simplified relaxed model without loosing any information, for the TIKHONOV regularization.
- **Aim 2):** improvement for the TOTAL VARIATION regularization, w.r.t. geodesic models, e.g. for the $\mathbb{S}_1$.

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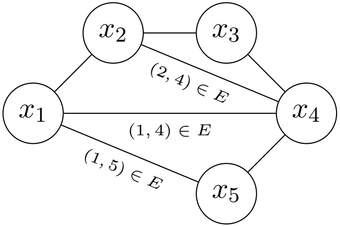

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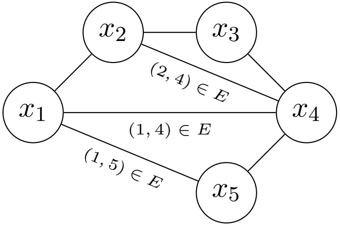

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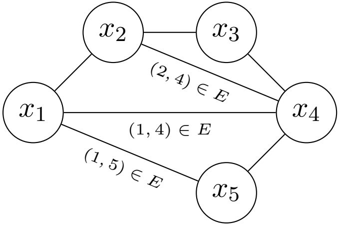

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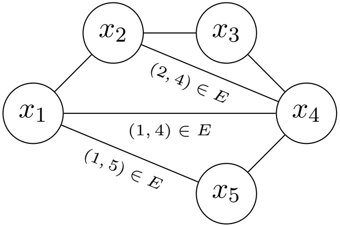

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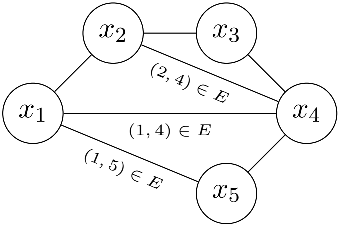

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- **Third approach:** application for TOTAL VARIATION regularization.

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- **Searching:** minimizer of the Tikhonov-like regularized functional (**nonconvex**)

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- **Exploiting:** $|x_n| = 1$, and rewriting the **nonconvex problem** as

$$(1) = \operatorname*{arg\,min}_{x\in\mathbb{S}_{\mathbb{C}}^N} - \sum_{n\in V} w_n \Re[x_n\bar{y}_n] - \sum_{(n,m)\in E} \lambda_{(n,m)} \Re[\bar{x}_m x_n]. \tag{2}$$

- Introducing $r \coloneqq (r_{(n,m)})_{(n,m)\in E} \in \mathbb{C}^M$, and rearranging last problem into

$$(1) = \operatorname*{arg\,min}_{x\in\mathbb{S}_{\mathbb{C}}^N, r\in\mathbb{C}^M} \mathcal{J}(x,r) \;\text{ s.t. }\; r_{(n,m)} = \bar{x}_m x_n \;\forall\, (n,m) \in E \tag{3}$$

with $\mathcal{J}(x,r) \coloneqq -\sum_{n\in V} w_n\, \Re[x_n \bar{y}_n] - \sum_{(n,m)\in E} \lambda_{(n,m)}\, \Re[r_{(n,m)}]$.

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Lemma 1

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$$P_{(n,m)} := \begin{bmatrix} 1 & x_n & x_m \\ \bar{x}_n & 1 & \bar{r}_{(n,m)} \\ \bar{x}_m & r_{(n,m)} & 1 \end{bmatrix} \in \mathbb{C}^{3\times 3} \tag{4}$$

is positive semi-definite and has rank one.

Related $\mathbb{R}^2$-Valued Model I

- **Aim:** generalization of the convex relaxation via

$$\mathbb{R}^2 \hookleftarrow \mathbb{S}_1 := \{\boldsymbol{z} \in \mathbb{R}^2 : \|\boldsymbol{z}\| = 1\} \simeq \mathbb{S}_{\mathbb{C}}.$$

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- Realizing the complex multiplication using the matrix representation of $z \in \mathbb{C}$

$$\boldsymbol{M}(z) := \boldsymbol{M}(\boldsymbol{z}) := \begin{bmatrix} \Re[\boldsymbol{z}] & -\Im[\boldsymbol{z}] \\ \Im[\boldsymbol{z}] & \Re[\boldsymbol{z}] \end{bmatrix} \Rightarrow \boldsymbol{M}(\boldsymbol{z})\boldsymbol{x} = \overrightarrow{zx} = \begin{bmatrix} \Re[zx] \\ \Im[zx] \end{bmatrix}. \tag{6}$$

Related $\mathbb{R}^2$-Valued Model II

– **Introducing:** $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{x}_n \in \mathbb{R}^2$, yields the real of the complex-valued version:

$$(7) = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N, \boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \;\text{ s.t. }\; \boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{x}_n \quad \forall (n,m) \in E$$

with $\mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \coloneqq -\sum_{n\in V} w_n \langle \boldsymbol{x}_n, \boldsymbol{y}_n \rangle - \sum_{(n,m)\in E} \lambda_{(n,m)} \Re[\boldsymbol{r}_{(n,m)}]$.

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Lemma 2 (Beinert, Bresch, Steidl [2])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_1$ and $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}}\boldsymbol{x}_n$ if and only if the block matrix

$$\boldsymbol{P}_{(n,m)} \coloneqq \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{x}_n) & \boldsymbol{M}(\boldsymbol{x}_m) \\ \boldsymbol{M}(\boldsymbol{x}_n)^{\mathrm{T}} & \boldsymbol{I}_2 & \boldsymbol{M}(\boldsymbol{r}_{(n,m)})^{\mathrm{T}} \\ \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} & \boldsymbol{M}(\boldsymbol{r}_{(n,m)}) & \boldsymbol{I}_2 \end{bmatrix} \in \mathbb{R}^{6\times 6} \tag{7}$$

is positive semi-definite and has rank two.

Simplified $\mathbb{R}^2$-Valued Model

– **Note:** second components of $\boldsymbol{r}_{(n,m)} = \boldsymbol{M}(\boldsymbol{x}_m)^{\mathrm{T}} \boldsymbol{x}_n$ in the **relaxed real model** originate form the complex-valued model.

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- **Claim:** equivalent real-valued problem

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_1^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m) \in E \tag{8}$$

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LEMMA 3 (BEINERT, BRESCH, STEIDL [2])

Let $n,m\in V$ and $(n,m)\in E$. Then $\boldsymbol{x}_n,\boldsymbol{x}_m\in\mathbb{S}_1$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m\rangle$ if and only if

$$\boldsymbol{Q}_{(n,m)} \succeq 0 \text{ and } \operatorname{rk}(\boldsymbol{Q}_{(n,m)}) = 2, \text{ where } \boldsymbol{Q}_{(n,m)} \coloneqq \begin{bmatrix} \boldsymbol{I}_2 & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \boldsymbol{\ell}_{(n,m)} \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{\ell}_{(n,m)} & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}. \tag{9}$$

Simplified $\mathbb{R}^2$-Valued Model

– **Again:** neglect the rank-two constraint, we propose **our simplified relaxed real model**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m) \in E. \tag{10}$$

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- Our convex model is simpler w.r.t. to the dimension of the matrix representation than
$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \quad \text{s.t.} \quad \boldsymbol{P}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E, \tag{11}$$

Theorem 4 (Beinert, Bresch, Steidl [2])

Both models are equivalent in the following sense:

(i) If $(\hat{\boldsymbol{x}},\hat{\boldsymbol{r}})$ solves (11), then $(\hat{\boldsymbol{x}},\Re[\hat{\boldsymbol{r}}])$ solves (10).

(ii) If $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{\ell}})$ solves (10), then $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{r}})$ with $\tilde{\boldsymbol{r}}_{(n,m)} = (\tilde{\boldsymbol{\ell}}_{(n,m)}, \Im[\boldsymbol{M}(\tilde{\boldsymbol{x}}_m)^{\mathrm{T}}\tilde{\boldsymbol{x}}_n])^{\mathrm{T}}$ solves (11).

Simplified $\mathbb{R}^2$-Valued Model

- **Again:** neglect the rank-two constraint, we propose **our simplified relaxed real model**:
$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{\ell}\in\mathbb{R}^M} \mathcal{K}(\boldsymbol{x},\boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{Q}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E. \tag{10}$$
- Our convex model is simpler w.r.t. to the dimension of the matrix representation than
$$\operatorname*{arg\,min}_{\boldsymbol{x}\in(\mathbb{R}^2)^N,\boldsymbol{r}\in(\mathbb{R}^2)^M} \mathcal{J}(\boldsymbol{x},\boldsymbol{r}) \quad \text{s.t.} \quad \boldsymbol{P}_{(n,m)} \succeq 0 \quad \forall (n,m)\in E, \tag{11}$$

Theorem 4 (Beinert, Bresch, Steidl [2])

Both models are equivalent in the following sense:

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Corollary 5

*The **simplified relaxed real model** and the **relaxed complex model** are equivalent.*

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- **Main advantage:** the **simplified relaxed real model** is simple to generalize to $(d-1)$-dimensions; let $\mathbb{S}_{d-1} := \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| = 1\}$.

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$$\operatorname*{arg\,min}_{\boldsymbol{x} \in \mathbb{S}_{d-1}^N} \sum_{n \in V} \frac{w_n}{2} \|\boldsymbol{x}_n - \boldsymbol{y}_n\|^2 + \sum_{(n,m) \in E} \frac{\lambda_{(n,m)}}{2} \|\boldsymbol{x}_n - \boldsymbol{x}_m\|^2 \tag{12}$$

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$$= \underset{\boldsymbol{x} \in \mathbb{S}_{d-1}^N, \boldsymbol{\ell} \in \mathbb{R}^M}{\arg\min} \mathcal{K}(\boldsymbol{x}, \boldsymbol{\ell}) \quad \text{s.t.} \quad \boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle \quad \forall (n,m) \in E \tag{13}$$

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Lemma 6 (Beinert, Bresch, Steidl [2])

Let $n, m \in V$ and $(n,m) \in E$. Then $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{S}_{d-1}$ and $\boldsymbol{\ell}_{(n,m)} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ if and only if

$$\boldsymbol{Q}_{(n,m)} := \begin{bmatrix} \boldsymbol{I}_d & \boldsymbol{x}_n & \boldsymbol{x}_m \\ \boldsymbol{x}_n^{\mathrm{T}} & 1 & \boldsymbol{\ell}_{(n,m)} \\ \boldsymbol{x}_m^{\mathrm{T}} & \boldsymbol{\ell}_{(n,m)} & 1 \end{bmatrix} \in \mathbb{R}^{(d+2)\times(d+2)} \tag{14}$$

is positive semi-definite and has rank d.

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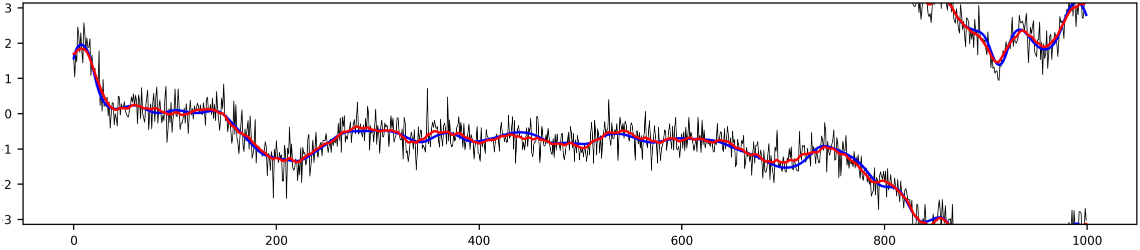
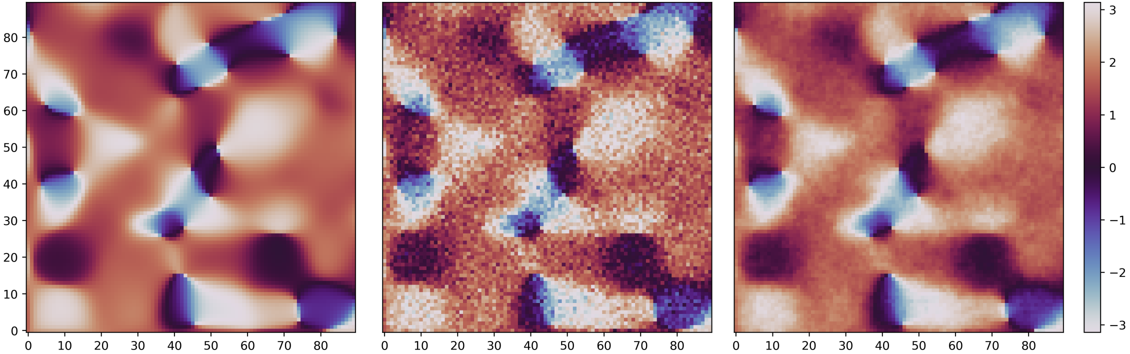


- synthetic, smooth signal $(\boldsymbol{x}_n)_{n \in V}$ on the line graph.
- noisy observation $(\boldsymbol{y}_n)_{n \in V}$ are generated using the *von Mises–Fisher* distribution by $\boldsymbol{y}_n \sim \mathcal{N}_{\mathrm{vMF}}(\boldsymbol{x}_n, \kappa)$ for all $n \in V$, where $\kappa > 0$ is the capacity, here $\kappa = 10$.

	mean	time	iter.	parameters	reg. parameters
PMM on **relaxed complex**	10^{-9}	2.10	201	$\tau = 0.1, \sigma = (4\tau)^{-1}$	
ADMM on **relaxed real**	10^{-12}	1.83	182	$\rho = 3$	$w_n = 1, \lambda_{(n,m)} = 25$
ADMM on **simp. relaxed real**	10^{-13}	1.77	181	$\rho = 3$	



	mean	time	iter.	parameters	reg. parameters
PMM on **relaxed complex**	10^{-4}	747.18	5536	$\tau = 0.1, \sigma = (8\tau)^{-1}$	
ADMM on **relaxed real**	10^{-4}	389.01	2457	$\rho = 20$	$w_n = 1, \lambda_{(n,m)} = 1$
ADMM on **simp. relaxed real**	10^{-4}	301.08	1943	$\rho = 20$	

Proof of the Concept III - Tikhonov Regularization

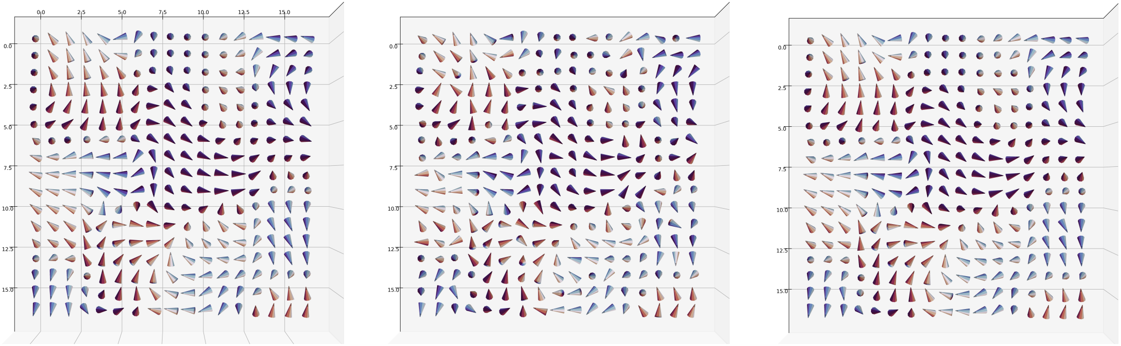


Figure 2: SO(3)-valued image graph: Ground truth (left), noisy observation with $\kappa_1 := 30$ and $\kappa_2 := 5$ (middle), and denoised version (right) The rotations are visualized by operating on a colored 3d cone. The entire signal consists of 90×90 pixel, where only every third pixel from the 25th to the 75th pixel is visualized. The parameters have been $w_n := 1$, $\lambda_{(n,m)} := 1$, and $\rho := 3$.

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- For this reason, we **replace the squared 2-norm** of the regularizer in the previous part **by the 1-norm**
- This yields the TOTAL VARIATION (TV) regularization:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_{d-1}^N} \frac{1}{2}\sum_{n\in V}\|\boldsymbol{x}_n-\boldsymbol{y}_n\|_2^2+\lambda\mathrm{TV}(\boldsymbol{x}) \quad \text{with} \quad \mathrm{TV}(\boldsymbol{x}) \coloneqq \sum_{(n,m)\in E}\|\boldsymbol{x}_n-\boldsymbol{x}_m\|_1. \tag{15}$$

Convexification

- Using the property of the squared-L_2 norm, we rewrite the objective

$$\mathcal{K}(\boldsymbol{x}) := -\sum_{n\in V}\langle \boldsymbol{x}_n, \boldsymbol{y}_n\rangle + \lambda \mathrm{TV}(\boldsymbol{x}) \qquad \text{with} \qquad (15) = \operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{S}_{d-1}^N} \mathcal{K}(\boldsymbol{x}).$$

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- Convexifying the sphere-valued domain $\mathbb{S}_{d-1}^N$, we propose **our relaxed convex problem**:

$$\operatorname*{arg\,min}_{\boldsymbol{x}\in\mathbb{R}^{d\times N}} \quad \mathcal{K}(\boldsymbol{x}) \quad \text{s.t.} \quad \boldsymbol{x}_n \in \mathbb{B}_d \quad \text{for all} \quad n \in V. \tag{16}$$

where $\mathbb{B}_d := \{\boldsymbol{\xi} \in \mathbb{R}^d : \|\boldsymbol{\xi}\|_2 \leq 1\}$.

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- Inserting $\boldsymbol{\ell}_{n,m} = \langle \boldsymbol{x}_n, \boldsymbol{x}_m \rangle$ and a Schur complement argument, yields the assertion.

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Lemma 7 (Beinert, B. [1])

For $\boldsymbol{x}_n, \boldsymbol{x}_m \in \mathbb{B}_1$, *it holds* $|\boldsymbol{x}_n - \boldsymbol{x}_m| = \frac{1}{2} \int_{-1}^{1} |\chi_\eta(\boldsymbol{x}_n) - \chi_\eta(\boldsymbol{x}_m)| \, \mathrm{d}\eta.$

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Theorem 8 (Beinert, B. [1])

Let $\boldsymbol{x}^* \in \mathbb{B}_1^N$ *be a solution of* (16) *for* $d = 1$. *Then,* $\chi_\eta(\boldsymbol{x}^*)$ *is a solution of* (15) *for almost all* $\eta \in [-1, 1]$.

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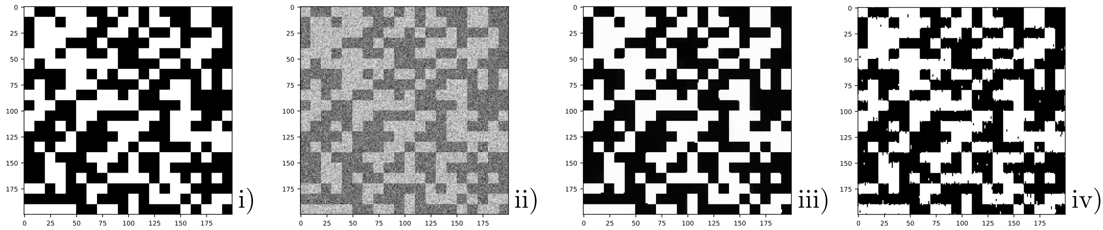
Table 1: Averages for 50 randomly generated QR codes for different noise levels. One specific instance is illustrated in Fig. 3 (fast-TV [5], ANISO-TV [4]).

$\sigma =$	Algorithm	signal errors MSE	signal errors MIoU	λ	time (sec.)	distance to sphere
$\sqrt{2}\frac{5}{10}$	fast-TV	**0.00036**	**0.99734**	0.7	< 0.1	0.07879
	ANISO-TV	0.00112	0.97495	1.9	1.8	0.00122
	ADMM-TV	**0.00036**	**0.99734**	0.7	1.1	**0.00000**
$\sqrt{2}\frac{7}{10}$	fast-TV	0.00069	0.99018	1.0	< 0.1	0.10021
	ANISO-TV	0.00190	0.92942	2.4	1.9	0.00732
	ADMM-TV	**0.00069**	**0.99030**	1.0	1.1	**0.00000**
$\sqrt{2}\frac{9}{10}$	fast-TV	0.00106	0.97741	1.7	< 0.1	0.12263
	ANISO-TV	0.00263	0.86956	2.9	2.1	0.01445
	ADMM-TV	**0.00106**	**0.97753**	1.7	1.2	**0.00000**

Table 2: Averages for 20 randomly generated noisy instances of the ground truth in Fig. 4 for different noise levels. (fast TV [5], CPPA-TV [3])

$\kappa =$	Algorithm	signal error MSE	λ	time (sec.)	distance to sphere
50	CPPA-TV	0.00076	0.15	128.2	—
	fast-TV	0.00056	0.25	< 0.1	0.00055
	ADMM-TV	**0.00043**	0.2	4.3	**0.00000**
20	CPPA-TV	0.00198	0.25	166.1	—
	fast-TV	0.00199	0.25	< 0.1	0.00098
	ADMM-TV	**0.00107**	0.3	6.2	**0.00000**
10	CPPA-TV	0.00409	0.55	191.7	—
	fast-TV	0.00388	0.25	< 0.1	0.00155
	ADMM-TV	**0.00218**	0.45	10.5	**0.00000**



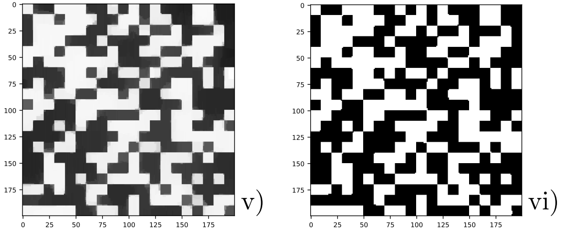


Figure 3: QR code denoising example: (i) ground truth, (ii) noisy data ($\sigma = \sqrt{2} \cdot 0.5$), (iii) ADMM-TV ($\lambda = 1.0$, $\rho = 0.1$) without projection, (iv) ANISO-TV ($\lambda = 1.6$) with projection χ_0, (v) fast-TV ($\lambda = 1.0$) without projection, (vi) fast-TV ($\lambda = 1.0$) with projection χ_0.

Comparison of $\mathbb{S}_1$-Images - TV Regularization

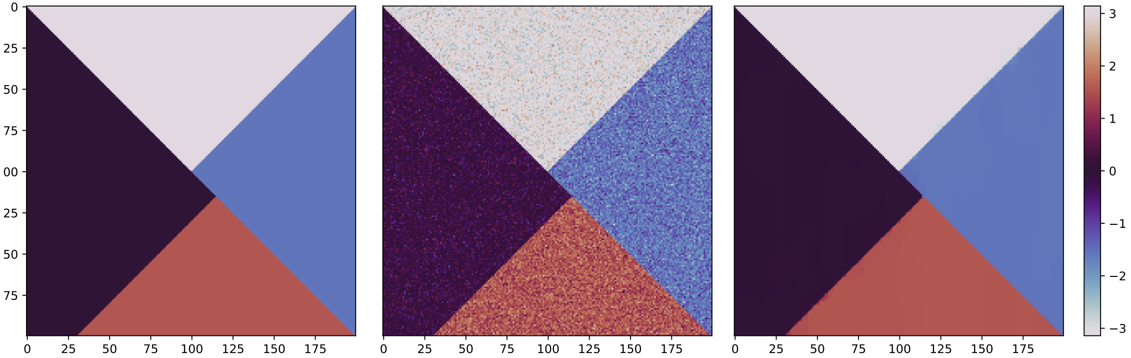


Figure 4: Toy-data example following [7] for $\mathbb{S}_1$-image denoising (from left to right): (i) ground truth, (ii) noisy measurement generated by the von Mises–Fisher distribution with $\kappa = 10$, (iii) solution via ADMM-TV ($\lambda = 0.55$, $\rho = 10$) without final projection.

Thank you for your attention!

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