Finitary $\mathcal{M}$-Adhesive Categories

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Abstract. Finitary $\mathcal{M}$-adhesive categories are $\mathcal{M}$-adhesive categories with finite objects only, where the notion $\mathcal{M}$-adhesive category is short for weak adhesive high-level replacement (HLR) category. We call an object finite if it has a finite number of $\mathcal{M}$-subobjects. In this paper, we show that in finitary $\mathcal{M}$-adhesive categories we do not only have all the well-known properties of $\mathcal{M}$-adhesive categories, but also all the additional HLR-requirements which are needed to prove the classical results for $\mathcal{M}$-adhesive systems. These results are the Local Church-Rosser, Parallelism, Concurrency, Embedding, Extension, and Local Confluence Theorems, where the latter is based on critical pairs. More precisely, we are able to show that finitary $\mathcal{M}$-adhesive categories have a unique $\mathcal{E}$-$\mathcal{M}$ factorization and initial pushouts, and the existence of an $\mathcal{M}$-initial object implies in addition finite coproducts and a unique $\mathcal{E}'$-$\mathcal{M}'$ pair factorization. Moreover, we can show that the finitary restriction of each $\mathcal{M}$-adhesive category is a finitary $\mathcal{M}$-adhesive category and finitariness is preserved under functor and comma category constructions based on $\mathcal{M}$-adhesive categories. This means that all the classical results are also valid for corresponding finitary $\mathcal{M}$-adhesive systems like several kinds of finitary graph and Petri net transformation systems. Finally, we discuss how some of the results can be extended to non-$\mathcal{M}$-adhesive categories.

1 Introduction

The concepts of adhesive [1] and (weak) adhesive high-level-replacement (HLR) [2] categories have been a break-through for the double pushout approach (DPO) of algebraic graph transformations [3]. Almost all main results in the DPO-approach have been formulated and proven in these categorical frameworks and instantiated to a large variety of HLR systems, including different kinds of graph and Petri net transformation systems. These main results include the Local Church-Rosser, Parallelism, and Concurrency Theorems, the Embedding and Extension Theorem, completeness of critical pairs, and the Local Confluence Theorem.

However, in addition to the well-known properties of adhesive and (weak) adhesive HLR categories $(\mathbf{C}, \mathcal{M})$, also the following additional HLR-requirements have been needed in [2] to prove these main results: finite coproducts compatible with $\mathcal{M}$, $\mathcal{E}'$-$\mathcal{M}'$ pair factorization usually based on suitable $\mathcal{E}$-$\mathcal{M}$ factorization

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of morphisms, and initial pushouts. It is an open question up to now under which conditions these additional HLR-requirements are valid in order to avoid an explicit verification for each instantiation of an adhesive or (weak) adhesive HLR category. In [4], this has been investigated for comma and functor category constructions of weak adhesive HLR categories, but the results hold only under strong preconditions. In this paper, we close this gap showing that these additional properties are valid in finitary $\mathcal{M}$-adhesive categories. We use the notion "$\mathcal{M}$-adhesive category" as short hand for "weak adhesive HLR category" in the sense of [2]. Moreover, an object A in an $\mathcal{M}$-adhesive category is called finite, if A has (up to isomorphism) only a finite number of $\mathcal{M}$-subobjects, i. e., only finite many $\mathcal{M}$-morphisms $m\colon A' \to A$ up to isomorphism. The category $\mathbf{C}$ is called finitary, if it has only finite objects. Note, that the notion "finitary" depends on the class $\mathcal{M}$ of monomorphisms and "$\mathbf{C}$ being finitary" must not be confused with "$\mathbf{C}$ being finite" in the sense of a finite number of objects and morphisms. In the standard cases of **Sets** and **Graphs** where $\mathcal{M}$ is the class of all monomorphisms, finite objects are exactly finite sets and finite graphs, respectively.

Although in most application areas for the theory of graph transformations only finite graphs are considered, the theory has been developed for general graphs, including also infinite graphs, and it is implicitly assumed that the results can be restricted to finite graphs and to attributed graphs with finite graph part, while the data algebra may be infinite. Obviously, not only **Sets** and **Graphs** are adhesive categories but also the full subcategories $\mathbf{Sets}_{\text{fin}}$ of finite sets and $\mathbf{Graphs}_{\text{fin}}$ of finite graphs. But to our knowledge it is an open question whether for each adhesive category $\mathbf{C}$ also the restriction $\mathbf{C}_{\text{fin}}$ to finite objects is again an adhesive category. As far as we know this is true, if the inclusion functor $I\colon \mathbf{C}_{\text{fin}} \to \mathbf{C}$ preserves monomorphisms, but we are not aware of any adhesive category, where this property fails, or whether this can be shown in general. In this paper, we consider $\mathcal{M}$-adhesive categories $(\mathbf{C},\mathcal{M})$ with restriction to finite objects $(\mathbf{C}_{\text{fin}},\mathcal{M}_{\text{fin}})$, where $\mathcal{M}_{\text{fin}}$ is the restriction of $\mathcal{M}$ to morphisms between finite objects. In this case, the inclusion functor $I\colon \mathbf{C}_{\text{fin}} \to \mathbf{C}$ preserves $\mathcal{M}$-morphisms, such that finite objects in $\mathbf{C}_{\text{fin}}$ w. r. t. $\mathcal{M}_{\text{fin}}$ are exactly the finite objects in $\mathbf{C}$ w. r. t. $\mathcal{M}$. More generally, we are able to show that the finitary restriction $(\mathbf{C}_{\text{fin}},\mathcal{M}_{\text{fin}})$ of any $\mathcal{M}$-adhesive category $(\mathbf{C},\mathcal{M})$ is a finitary $\mathcal{M}$-adhesive category. Moreover, finitariness is preserved under functor and comma category constructions based on $\mathcal{M}$-adhesive categories.

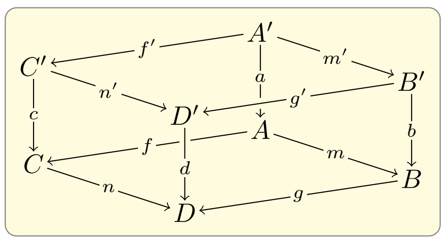
In Section 2, we introduce basic notions of finitary $\mathcal{M}$-adhesive categories including finite coproducts compatible with $\mathcal{M}$, $\mathcal{M}$-initial objects, finite objects, and finite intersections, which are essential for the theory of finitary $\mathcal{M}$-adhesive categories. The first main result, showing that the additional HLR-requirements mentioned above are valid for finitary $\mathcal{M}$-adhesive categories, is presented in Section 3. In Section 4, we show as second main result that the finitary restriction of an $\mathcal{M}$-adhesive category is a finitary $\mathcal{M}$-adhesive category such that the results of Section 3 are applicable. In Section 5, we show that functorial constructions, including functor and comma categories, applied to finitary $\mathcal{M}$-adhesive categories are again finitary $\mathcal{M}$-adhesive categories under suitable conditions. In Section 6, we analyze how some of the results in Section 3 can be shown in a weaker form for (finitary) non-$\mathcal{M}$-adhesive categories, like the category of simple graphs with all monomorphisms $\mathcal{M}$. Especially, we consider the construction of weak initial pushouts which are the basis for the gluing condition in order to construct (unique) minimal pushout complements in such categories, while initial pushouts are the basis for the construction of (unique) pushout complements in (finitary) $\mathcal{M}$-adhesive categories. In the conclusion, we summarize the main results and discuss open problems for future research. The full proofs for all propositions can be found in the unabridged technical report [5].

2 Basic Notions of Finitary $\mathcal{M}$-Adhesive Categories

Adhesive categories have been introduced by Lack and Sobociński in [1] and generalized to (weak) adhesive HLR categories in [6, 2] as a categorical framework for various kinds of graph and net transformation systems.

An *$\mathcal{M}$-adhesive category* $(\mathbf{C}, \mathcal{M})$, called weak adhesive HLR category in [2], consists of a category $\mathbf{C}$ and a class $\mathcal{M}$ of monomorphisms in $\mathbf{C}$, which is closed under isomorphisms, composition, and decomposition ($g \circ f \in \mathcal{M}$ and $g \in \mathcal{M}$ implies $f \in \mathcal{M}$), such that $\mathbf{C}$ has pushouts and pullbacks along $\mathcal{M}$-morphisms, $\mathcal{M}$-morphisms are closed under pushouts and pullbacks, and pushouts along $\mathcal{M}$-morphisms are weak van Kampen (VK) squares.

A *weak VK square* is a pushout as at the bottom of the cube in the adjacent figure with $m \in \mathcal{M}$, which satisfies the *weak VK property*, i. e., for any commutative cube, where the back faces are pullbacks and ($f \in \mathcal{M}$ or $b, c, d \in \mathcal{M}$), the following statement holds: The top face is a pushout if and only if the front faces are pullbacks. In contrast, the (non-weak) VK property does not assume ($f \in \mathcal{M}$ or $b, c, d \in \mathcal{M}$).



Well-known examples of $\mathcal{M}$-adhesive categories are the categories $(\mathbf{Sets}, \mathcal{M})$ of sets, $(\mathbf{Graphs}, \mathcal{M})$ of graphs, $(\mathbf{Graphs}_{TG}, \mathcal{M})$ of typed graphs, $(\mathbf{ElemNets}, \mathcal{M})$ of elementary Petri nets, $(\mathbf{PTNets}, \mathcal{M})$ of place/transition nets, where for all these categories $\mathcal{M}$ is the class of all monomorphisms, and $(\mathbf{AGraphs}_{ATG}, \mathcal{M})$ of typed attributed graphs, where $\mathcal{M}$ is the class of all injective typed attributed graph morphisms with isomorphic data type component (see [2]).

The compatibility of the morphism class $\mathcal{M}$ with (finite) coproducts was required for the construction of parallel rules in [2], but in fact finite coproducts (if they exist) are always compatible with $\mathcal{M}$ in $\mathcal{M}$-adhesive categories.

Proposition 1 (Finite Coproducts Compatible with $\mathcal{M}$). *For each $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ with finite coproducts, finite coproducts are compatible with $\mathcal{M}$, i. e., $f_i \in \mathcal{M}$ for $i = 1, \dots, n$ implies that $f_1 + \dots + f_n \in \mathcal{M}$.*

Proof (Idea). The proof constructs a coproduct $f+g\colon A+B \to A'+B'$ of morphisms as the composition of $f+\mathrm{id}_B$ and $\mathrm{id}_{A'}+g$ resulting from pushouts. Since $\mathcal{M}$ is closed under pushouts and composition this implies $f+g \in \mathcal{M}$. $\square$

For the construction of coproducts, it often makes sense to use pushouts over $\mathcal{M}$-initial objects in the following sense.

Definition 1 ($\mathcal{M}$-Initial Object). *An initial object I in $(\mathbf{C},\mathcal{M})$ is called* $\mathcal{M}$-initial *if for each object $A \in \mathbf{C}$ the unique morphism $i_A\colon I \to A$ is in $\mathcal{M}$.*

Note that if $(\mathbf{C},\mathcal{M})$ has an $\mathcal{M}$-initial object then all initial objects are $\mathcal{M}$-initial due to $\mathcal{M}$ being closed under isomorphisms and composition.

In the $\mathcal{M}$-adhesive categories $(\mathbf{Sets},\mathcal{M})$, $(\mathbf{Graphs},\mathcal{M})$, $(\mathbf{Graphs}_{TG},\mathcal{M})$, $(\mathbf{ElemNets},\mathcal{M})$, and $(\mathbf{PTNets},\mathcal{M})$ we have $\mathcal{M}$-initial objects defined by the empty set, empty graphs, and empty nets, respectively. But in $(\mathbf{AGraphs}_{ATG}, \mathcal{M})$, there is no $\mathcal{M}$-initial object. The initial attributed graph $(\varnothing, T_{DSIG})$ with term algebra T_{DSIG} of the data type signature $DSIG$ is not $\mathcal{M}$-initial because the data type part of the unique morphism $(\varnothing, T_{DSIG}) \to (G, D)$ is, in general, not an isomorphism.

The existence of an $\mathcal{M}$-initial object implies that we have finite coproducts.

Proposition 2 (Existence of Finite Coproducts). *For each $\mathcal{M}$-adhesive category $(\mathbf{C},\mathcal{M})$ with $\mathcal{M}$-initial object, $(\mathbf{C},\mathcal{M})$ has finite coproducts, where the injections into coproducts are in $\mathcal{M}$.*

Proof (Idea). Coproducts can be constructed as pushouts under the $\mathcal{M}$-initial object. Since the morphisms from the $\mathcal{M}$-initial object are in $\mathcal{M}$ and $\mathcal{M}$ is closed under pushouts, the injections into the resulting coproduct are also in $\mathcal{M}$. $\square$

Note that an $\mathcal{M}$-adhesive category may still have coproducts even if it does not have an $\mathcal{M}$-initial object. The $\mathcal{M}$-adhesive category $(\mathbf{AGraphs}_{ATG},\mathcal{M})$, e. g., has finite coproducts as shown in [2].

$$\begin{tikzcd} A_1' \arrow[drr, "m_1"] \arrow[d, "i"'] \arrow[dr, phantom, "(=)"] & & \\ A_2' \arrow[rr, "m_2"'] & & A \end{tikzcd}$$

Now, we are going to consider finite objects in $\mathcal{M}$-adhesive categories. Intuitively, we are interested in those objects where the graph or net part is finite. This can be expressed in a general $\mathcal{M}$-adhesive category by the fact that we have only a finite number of $\mathcal{M}$-subobjects. An *$\mathcal{M}$-subobject* of an object A is an isomorphism class of $\mathcal{M}$-morphisms $m\colon A' \to A$, where $\mathcal{M}$-morphisms $m_1\colon A_1' \to A$ and $m_2\colon A_2' \to A$ belong to the same $\mathcal{M}$-subobject of A if there is an isomorphism $i\colon A_1' \xrightarrow{\sim} A_2'$ with $m_1 = m_2 \circ i$.

Definition 2 (Finite Object and Finitary $\mathcal{M}$-Adhesive Category). *An object A in an $\mathcal{M}$-adhesive category $(\mathbf{C},\mathcal{M})$ is called* finite *if A has finitely many $\mathcal{M}$-subobjects.*

An $\mathcal{M}$-adhesive category $(\mathbf{C},\mathcal{M})$ is called finitary, *if each object $A \in \mathbf{C}$ is finite.*

In $(\mathbf{Sets},\mathcal{M})$, the finite objects are the finite sets. Graphs in $(\mathbf{Graphs},\mathcal{M})$ and $(\mathbf{Graphs}_{TG},\mathcal{M})$ are finite if the node and edge sets have finite cardinality, while

TG itself may be infinite. Petri nets in $(\mathbf{ElemNets}, \mathcal{M})$ and $(\mathbf{PTNets}, \mathcal{M})$ are finite if the number of places and transitions is finite. A typed attributed graph $AG = ((G, D), t)$ in $(\mathbf{AGraphs}_{ATG}, \mathcal{M})$ with typing $t\colon (G, D) \to ATG$ is finite if the graph part of G, i. e., all vertex and edge sets except the set V_D of data vertices generated from D, is finite, while the attributed type graph ATG or the data type part D may be infinite, because $\mathcal{M}$-morphisms are isomorphisms on the data type part.

In the following, we will use finite $\mathcal{M}$-intersections in various constructions. Finite $\mathcal{M}$-intersections are a generalization of pullbacks to an arbitrary, but finite number of $\mathcal{M}$-subobjects and, thus, a special case of limits.

Definition 3 (Finite $\mathcal{M}$-Intersection). *Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ and morphisms $m_i\colon A_i \to B \in \mathcal{M}$ ($i \in \mathcal{I}$ for finite $\mathcal{I}$) with the same codomain object B, a* finite $\mathcal{M}$-intersection *of m_i ($i \in \mathcal{I}$) is an object A with morphisms $n_i\colon A \to A_i$ ($i \in \mathcal{I}$), such that $m_i \circ n_i = m_j \circ n_j$ ($i, j \in \mathcal{I}$) and for each other object A' and morphisms $n'_i\colon A' \to A_i$ ($i \in \mathcal{I}$) with $m_i \circ n'_i = m_j \circ n'_j$ ($i, j \in \mathcal{I}$) there is a unique morphism $a\colon A' \to A$ with $n_i \circ a = n'_i$ ($i \in \mathcal{I}$).*

$$
\begin{tikzcd}
 & & A_i \arrow[dr, "m_i"] & \\
A' \arrow[urr, bend left, "n'_i"] \arrow[drr, bend right, "n'_j"'] \arrow[r, "\exists! a"] \arrow[ur, phantom, "(=)"] \arrow[dr, phantom, "(=)"] & A \arrow[ur, "n_i"] \arrow[dr, "n_j"'] & (=) & B \\
 & & A_j \arrow[ur, "m_j"'] &
\end{tikzcd}
$$

Note that finite $\mathcal{M}$-intersections can be constructed by iterated pullbacks and, hence, always exist in $\mathcal{M}$-adhesive categories. Moreover, since pullbacks preserve $\mathcal{M}$-morphisms, the morphisms n_i are also in $\mathcal{M}$.

3 Additional HLR-Requirements for Finitary $\mathcal{M}$-adhesive Categories

In order to prove the main classical results for $\mathcal{M}$-adhesive systems based on $\mathcal{M}$-adhesive categories additional HLR-requirements have been used in [2]. For the Parallelism Theorem, binary coproducts compatible with $\mathcal{M}$ are required in order to construct parallel rules. Initial pushouts are used in order to define the gluing condition and to show that consistency in the Embedding Theorem is not only sufficient, but also necessary. In connection with the Concurrency Theorem and for completeness of critical pairs, an $\mathcal{E}'$-$\mathcal{M}'$ pair factorization is used such that the class $\mathcal{M}'$ satisfies the $\mathcal{M}$-$\mathcal{M}'$ pushout-pullback decomposition property. Moreover, a standard construction for $\mathcal{E}'$-$\mathcal{M}'$ pair factorization uses an $\mathcal{E}$-$\mathcal{M}$ factorization of morphisms in $\mathbf{C}$, where $\mathcal{E}'$ is constructed from $\mathcal{E}$ using binary coproducts.

As far as we know, these additional HLR-requirements cannot be concluded from the axioms of $\mathcal{M}$-adhesive categories, at least we do not know proofs for non-trivial classes $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{M}$, and $\mathcal{M}'$. However, in the case of finitary $\mathcal{M}$-adhesive categories $(\mathbf{C}, \mathcal{M})$ we are able to show that these additional HLR-requirements are valid for suitable classes $\mathcal{E}$ and $\mathcal{E}'$, and $\mathcal{M}' = \mathcal{M}$. Note that for $\mathcal{M}' = \mathcal{M}$, the $\mathcal{M}$-$\mathcal{M}'$ pushout-pullback decomposition property is the $\mathcal{M}$ pushout-pullback decomposition property which is valid already in general $\mathcal{M}$-adhesive categories.

The reason for the existence of an $\mathcal{E}$-$\mathcal{M}$ factorization of morphisms in finitary $\mathcal{M}$-adhesive categories is the fact that we only need finite intersections of $\mathcal{M}$-subobjects and not infinite intersections as would be required in general $\mathcal{M}$-adhesive categories. Moreover, we fix the choice of the class $\mathcal{E}$ to extremal morphisms w. r. t. $\mathcal{M}$.

The dependencies are shown in Fig. 1, where the additional assumptions of finitariness and $\mathcal{M}$-initial objects are shown in the top row, the HLR-requirements shown in this paper in the center and the classical theorems in the bottom row.

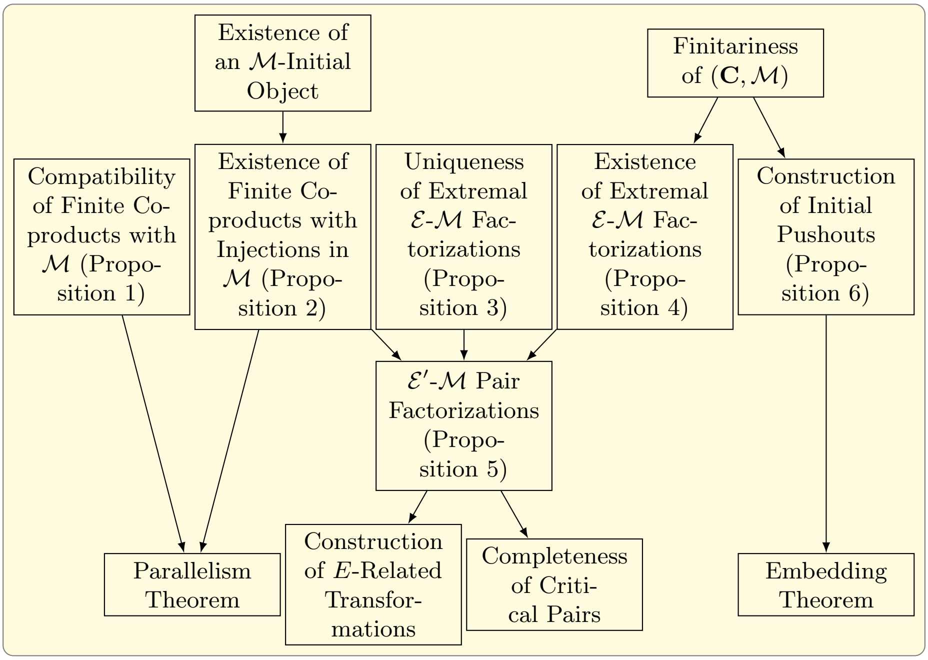


Fig. 1. Dependency graph

Definition 4 (Extremal $\mathcal{E}$-$\mathcal{M}$ Factorization). *Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, the class $\mathcal{E}$ of all* extremal morphisms w. r. t. $\mathcal{M}$ *is defined by* $\mathcal{E} := \{e \textit{ in } \mathbf{C} \mid \textit{for all } m, f \textit{ in } \mathbf{C} \textit{ with } m \circ f = e\colon\ m \in \mathcal{M} \textit{ implies } m \textit{ isomorphism}\}$. *For a morphism $f\colon A \to B$ in $\mathbf{C}$ an* extremal $\mathcal{E}$-$\mathcal{M}$ factorization *of f is given by an object $\bar{B}$ and morphisms $e\colon A \to \bar{B} \in \mathcal{E}$ and $m\colon \bar{B} \to B \in \mathcal{M}$, such that $m \circ e = f$.*

Remark 1. Although in several example categories the class $\mathcal{E}$ consists of all epimorphisms, we will show below that the class $\mathcal{E}$ of extremal morphisms w. r. t. $\mathcal{M}$ is not necessarily a class of epimorphisms. But if we require $\mathcal{M}$ to be the class of all monomorphisms and e and f in the definition of $\mathcal{E}$ in Definition 4 to be epimorphisms then $\mathcal{E}$ is the class of all extremal epimorphisms in the sense of [7].

Proposition 3 (Uniqueness of Extremal $\mathcal{E}$-$\mathcal{M}$ Factorizations). *Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, then extremal $\mathcal{E}$-$\mathcal{M}$ factorizations are unique up to isomorphism.*

Proof (Idea). For two extremal $\mathcal{E}$-$\mathcal{M}$ factorizations of a morphism $f\colon A \to B$, we construct a pullback over the two morphisms in $\mathcal{M}$ with the resulting morphisms also in $\mathcal{M}$. The universal property of the pullback induces a unique morphism from A into the pullback object which, together with the pullback morphisms, factors the two morphisms in $\mathcal{E}$. Since these are extremal, the pullback morphisms are isomorphisms and the two extremal $\mathcal{E}$-$\mathcal{M}$ factorizations are isomorphic. $\square$

Proposition 4 (Existence of Extremal $\mathcal{E}$-$\mathcal{M}$ Factorizations). *Given a finitary $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, then we can construct an extremal $\mathcal{E}$-$\mathcal{M}$ factorization $m \circ e = f$ for each morphism $f\colon A \to B$ in $\mathbf{C}$.*

Construction: $m\colon \bar{B} \to B$ is constructed as $\mathcal{M}$-intersection of all $\mathcal{M}$-subobjects $m_i\colon B_i \to B$ for which there exists $e_i\colon A \to B_i$ with $f = m_i \circ e_i$, leading to a suitable finite index set $\mathcal{I}$, and $e\colon A \to \bar{B}$ is the induced unique morphism with $\bar{m}_i \circ e = e_i$ for all $i \in \mathcal{I}$.

$$\begin{tikzcd}
 & & A \arrow[dll, bend right, "e"'] \arrow[dl, "e_i"'] \arrow[dd, "f"] \\
\bar{B} \arrow[r, "\bar{m}_i"] \arrow[drr, bend right, "m"'] & B_i \arrow[dr, "m_i"] \arrow[u, phantom, "(=)"] \arrow[d, phantom, "(=)"] \arrow[r, phantom, "(=)"] & \\
 & & B
\end{tikzcd}$$

Proof (Idea). The construction is always possible, since there is at least the trivial $\mathcal{M}$-subobject $m_i = \mathrm{id}_B$ with $e_i = f$ and at most finitely many $\mathcal{M}$-subobjects. It results in $\bar{m}_i \in \mathcal{M}$ and $m \in \mathcal{M}$, because $\mathcal{M}$ is closed under pullbacks and composition. The induced morphism e is in $\mathcal{E}$, since each factorization $e = m' \circ e'$ leads to another subobject $m_i = m \circ m'$ of B with $e_i = e'$ and the intersection induces an inverse of m'. $\square$

In the categories $(\mathbf{Sets}, \mathcal{M})$, $(\mathbf{Graphs}, \mathcal{M})$, $(\mathbf{Graphs_{TG}}, \mathcal{M})$, $(\mathbf{ElemNets}, \mathcal{M})$, and $(\mathbf{PTNets}, \mathcal{M})$, the extremal $\mathcal{E}$-$\mathcal{M}$ factorization $f = m \circ e$ for $f\colon A \to B$ with finite A and B is nothing else but the well-known epi-mono factorization of morphisms, which also works for infinite objects A and B, because these categories have not only finite but also general intersections. For $(\mathbf{AGraphs_{ATG}}, \mathcal{M})$, the extremal $\mathcal{E}$-$\mathcal{M}$ factorization of $(f_G, f_D)\colon (G, D) \to (G', D')$ with finite (or also infinite) G and G' is given by $(f_G, f_D) = (m_G, m_D) \circ (e_G, e_D)$ with $(e_G, e_D)\colon (G, D) \to (\bar{G}, \bar{D})$ and $(m_G, m_D)\colon (\bar{G}, \bar{D}) \to (G', D')$, where e_G is an epimorphism, m_G a monomorphism and m_D an isomorphism. In general, e_D and, hence, also (e_G, e_D) is not an epimorphism, since m_D is an isomorphism and, therefore, e_D has to be essentially the same as f_D. This means that the class $\mathcal{E}$, which depends on $\mathcal{M}$, is not necessarily a class of epimorphisms.

Given an $\mathcal{E}$-$\mathcal{M}'$ factorization and binary coproducts, we are able to construct an $\mathcal{E}'$-$\mathcal{M}'$ pair factorization in a standard way (see [2]), where we will consider the special case $\mathcal{M}' = \mathcal{M}$. First we recall $\mathcal{E}'$-$\mathcal{M}'$ pair factorizations.

Definition 5 ($\mathcal{E}'$-$\mathcal{M}'$ Pair Factorization). *Given a morphism class $\mathcal{M}'$ and a class $\mathcal{E}'$ of morphism pairs with common codomain in a category* $\mathbf{C}$, *then* $\mathbf{C}$ *has an $\mathcal{E}'$-$\mathcal{M}'$* pair factorization *if for each pair of morphisms $f_A\colon A \to D$, $f_B\colon B \to D$ there is, unique up to isomorphism, an object C and morphisms $e_A\colon A \to C$, $e_B\colon B \to C$, and $m\colon C \to D$ with $(e_A, e_B) \in \mathcal{E}'$, $m \in \mathcal{M}'$, $m \circ e_A = f_A$ and $m \circ e_B = f_B$.*

$$\begin{tikzcd}
A \arrow[dr, "e_A"'] \arrow[drr, "f_A"] & & \\
& C \arrow[r, tail, "m"] & D \\
B \arrow[ur, "e_B"] \arrow[urr, "f_B"'] & &
\end{tikzcd}$$

Proposition 5 (Construction of $\mathcal{E}'$-$\mathcal{M}'$ Pair Factorization). *Given a category* $\mathbf{C}$ *with an $\mathcal{E}$-$\mathcal{M}'$ factorization and binary coproducts, then* $\mathbf{C}$ *has also an $\mathcal{E}'$-$\mathcal{M}'$ pair factorization for the class $\mathcal{E}' = \{(e_A\colon A \to C, e_B\colon B \to C) \mid e_A, e_B \in \mathbf{C}$ with induced $e\colon A + B \to C \in \mathcal{E}\}$.*

Proof (Idea). For morphisms $f_A\colon A \to D$ and $f_B\colon B \to D$, we use the $\mathcal{E}$-$\mathcal{M}'$ factorization $f = m \circ e$ of the induced morphism $f\colon A + B \to D$ and obtain e_A and e_B by composing the respective coproduct inclusions with e. □

Remark 2. With the previous facts, we have extremal $\mathcal{E}$-$\mathcal{M}$ factorizations and corresponding $\mathcal{E}'$-$\mathcal{M}$ pair factorizations for all finitary $\mathcal{M}$-adhesive categories with $\mathcal{M}$-initial objects and these factorizations are unique up to isomorphism.

Finally, let us consider the construction of initial pushouts in finitary $\mathcal{M}$-adhesive categories. Similar to the extremal $\mathcal{E}$-$\mathcal{M}$ factorization, we are able to construct initial pushouts by finite $\mathcal{M}$-intersections of $\mathcal{M}$-subobjects in finitary $\mathcal{M}$-adhesive categories, but not in general ones. First we recall the definition.

Definition 6 (Initial Pushout). *A pushout* (1) *over a morphism $m\colon L \to G$ with $b, c \in \mathcal{M}$ in an $\mathcal{M}$-adhesive category* $(\mathbf{C}, \mathcal{M})$ *is called* initial *if the following condition holds: for all pushouts* (2) *over m with $b', c' \in \mathcal{M}$ there exist unique morphisms $b^*, c^* \in \mathcal{M}$ such that $b' \circ b^* = b$, $c' \circ c^* = c$, and* (3) *is a pushout.*

$$\begin{tikzcd}
B \arrow[r, tail, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \\
C \arrow[r, tail, "c"'] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, tail, "b^*"] \arrow[rr, tail, bend left, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(3)"] & B' \arrow[r, tail, "b'"] \arrow[d, "a'"] \arrow[dr, phantom, "(2)"] & L \arrow[d, "m"] \\
C \arrow[r, tail, "c^*"'] \arrow[rr, tail, bend right, "c"'] & C' \arrow[r, tail, "c'"'] & G
\end{tikzcd}$$

Remark 3. As shown in [2], the initial pushout allows to define a gluing condition, which is necessary and sufficient for the construction of pushout complements. Given $m\colon L \to G$ with initial pushout (1) and $l\colon K \to L \in \mathcal{M}$, which can be considered as the left-hand side of a rule, the gluing condition is satisfied if there exists $b^*\colon B \to K$ with $l \circ b^* = b$. In this case, the pushout complement object D in (2) is constructed as pushout object of a and b^*.

$$\begin{tikzcd}
B \arrow[r, tail, "b"] \arrow[rr, tail, bend left, "b^*"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \arrow[dr, phantom, "(2)"] & K \arrow[l, tail, "l"'] \arrow[d, "k"] \\
C \arrow[r, tail, "c"'] \arrow[rr, tail, bend right, "c^*"'] & G & D \arrow[l, tail, "f"]
\end{tikzcd}$$

Proposition 6 (Initial Pushouts in Finitary $\mathcal{M}$-Adhesive Categories). *Each finitary $\mathcal{M}$-adhesive category has initial pushouts.*

$$\begin{tikzcd} B \arrow[r, "u_i"] \arrow[rr, bend left, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(\mathrm{Q}_i)"] & B_i \arrow[r, "b_i"] \arrow[d, "a_i"] \arrow[dr, phantom, "(\mathrm{P}_i)"] & L \arrow[d, "m"] \\ C \arrow[r, "v_i"'] \arrow[rr, bend right, "c"'] & C_i \arrow[r, "c_i"'] & G \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \\ C \arrow[r, "c"'] & G \end{tikzcd}$$

Construction: Given $m: L \to G$*, we consider all those* $\mathcal{M}$*-subobjects* $b_i: B_i \to L$ *of* L *and* $c_i: C_i \to G$ *of* G *such that there is a pushout* (P_i) *over* m*. Since* L *and* G *are finite this leads to a finite index set* $\mathcal{I}$ *for all* (P_i) *with* $i \in \mathcal{I}$*. Now construct* $b: B \to L$ *as the finite* $\mathcal{M}$*-intersection of* $(b_i)_{i \in \mathcal{I}}$ *and* $c: C \to G$ *as the finite* $\mathcal{M}$*-intersection of* $(c_i)_{i \in \mathcal{I}}$*. Then there is a unique* $a: B \to C$ *such that* (Q_i) *commutes for all* $i \in \mathcal{I}$ *and the outer diagram* (1) *is the initial pushout over* m*.*

Proof (Idea). We have to show that (1) is a pushout. This is done by constructing the finite $\mathcal{M}$-intersections B and C by iterated pullbacks. In each step, the weak VK property is used to show that the pushouts are pulled back and composition of pushouts then leads to the diagonal square also being a pushout. The pushout (1) is also initial, since for each comparison pushout $(1')$ there is an $i_0 \in \mathcal{I}$ for which (P_{i_0}) is isomorphic to $(1')$ and the initiality property is given by the corresponding pushout (Q_{i_0}). □

The following theorem summarizes that the additional HLR-requirements mentioned above are valid for all finitary $\mathcal{M}$-adhesive categories.

Theorem 1 (Additional HLR-Requirements in Finitary $\mathcal{M}$-adhesive Categories). *Given a finitary* $\mathcal{M}$*-adhesive category* $(\mathbf{C}, \mathcal{M})$*, the following additional HLR-requirements are valid:*

1. $(\mathbf{C}, \mathcal{M})$ *has initial pushouts.*
2. $(\mathbf{C}, \mathcal{M})$ *has a unique extremal* $\mathcal{E}$*-*$\mathcal{M}$ *factorization, where* $\mathcal{E}$ *is the class of all extremal morphisms w. r. t.* $\mathcal{M}$*.*

If $(\mathbf{C}, \mathcal{M})$ *has an* $\mathcal{M}$*-initial object, we also have that:*

3. $(\mathbf{C}, \mathcal{M})$ *has finite coproducts compatible with* $\mathcal{M}$*.*
4. $(\mathbf{C}, \mathcal{M})$ *has a unique* $\mathcal{E}'$*-*$\mathcal{M}'$ *pair factorization for the classes* $\mathcal{M}' = \mathcal{M}$ *and* $\mathcal{E}'$ *induced by* $\mathcal{E}$*.*

Proof. Requirement 1 follows from Proposition 6, Requirement 2 from Propositions 3 and 4, Requirement 3 from Propositions 2 and 1, and, finally, Requirement 4 from Proposition 5. □

4 Finitary Restriction of $\mathcal{M}$-Adhesive Categories

In order to construct $\mathcal{M}$-adhesive categories it is important to know that (**Sets**, $\mathcal{M}$) is an $\mathcal{M}$-adhesive category, and that $\mathcal{M}$-adhesive categories are closed under product, slice, coslice, functor, and comma category constructions, provided that suitable conditions are satisfied (see [2]). This allows to show that $(\mathbf{Graphs}, \mathcal{M})$, $(\mathbf{Graphs_{TG}}, \mathcal{M})$, $(\mathbf{ElemNets}, \mathcal{M})$, and $(\mathbf{PTNets}, \mathcal{M})$ are also $\mathcal{M}$-adhesive categories. However, it is more difficult to show similar results for the additional HLR-requirements considered in Section 3, especially there are only weak results concerning the existence and construction of initial pushouts [4].

We have already shown that these additional HLR-requirements are valid in finitary $\mathcal{M}$-adhesive categories under weak assumptions. It remains to show how to construct finitary $\mathcal{M}$-adhesive categories. In the main result of this section, we show that for any $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ the restriction $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ to finite objects is a finitary $\mathcal{M}$-adhesive category, where $\mathcal{M}_{\text{fin}}$ is the corresponding restriction of $\mathcal{M}$. Moreover, we know how to construct pushouts and pullbacks in $\mathbf{C}_{\text{fin}}$ along $\mathcal{M}_{\text{fin}}$-morphisms, because the inclusion functor $I_{\text{fin}}\colon \mathbf{C}_{\text{fin}} \to \mathbf{C}$ creates and preserves pushouts and pullbacks along $\mathcal{M}_{\text{fin}}$ and $\mathcal{M}$, respectively.

Definition 7 (Finitary Restriction of $\mathcal{M}$-adhesive Category). *Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ the restriction to all finite objects of $(\mathbf{C}, \mathcal{M})$ defines the full subcategory $\mathbf{C}_{\text{fin}}$ of $\mathbf{C}$, and $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ with $\mathcal{M}_{\text{fin}} = \mathcal{M} \cap \mathbf{C}_{\text{fin}}$ is called* finitary restriction *of $(\mathbf{C}, \mathcal{M})$.*

Remark 4. Note, that an object A in $\mathbf{C}$ is finite in $(\mathbf{C}, \mathcal{M})$ if and only if A is finite in $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$. If $\mathcal{M}$ is the class of all monomorphisms in $\mathbf{C}$ then $\mathcal{M}_{\text{fin}}$ is not necessarily the class of all monomorphisms in $\mathbf{C}_{\text{fin}}$. This means that for an adhesive category $\mathbf{C}$, which is based on the class of all monomorphisms, there may be monomorphisms in $\mathbf{C}_{\text{fin}}$ which are not monomorphisms in $\mathbf{C}$, such that it is not clear whether the finite objects in $\mathbf{C}$ and $\mathbf{C}_{\text{fin}}$ are the same. This problem is avoided for $\mathcal{M}$-adhesive categories, where finitariness depends on $\mathcal{M}$.

In order to prove that with $(\mathbf{C}, \mathcal{M})$ also $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ is an $\mathcal{M}$-adhesive category, we have to analyze the construction and preservation of pushouts and pullbacks in $(\mathbf{C}, \mathcal{M})$ and $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$. This corresponds to the following creation and preservation properties of the inclusion functor $I_{\text{fin}}\colon \mathbf{C}_{\text{fin}} \to \mathbf{C}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "k"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

Definition 8 (Creation and Preservation of Pushout and Pullback). *Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$, the inclusion functor $I_{\text{fin}}\colon \mathbf{C}_{\text{fin}} \to \mathbf{C}$* creates pushouts along *$\mathcal{M}$ if for each pair of morphisms f, h in $\mathbf{C}_{\text{fin}}$ with $f \in \mathcal{M}_{\text{fin}}$ and pushout (1) in $\mathbf{C}$ we have already $D \in \mathbf{C}_{\text{fin}}$ such that (1) is a pushout in $\mathbf{C}_{\text{fin}}$ along $\mathcal{M}_{\text{fin}}$.*
Similarly, I_{fin} creates pullbacks along *$\mathcal{M}$ if for each pullback (1) in $\mathbf{C}$ with $g \in \mathcal{M}_{\text{fin}}$ and $B, C, D \in \mathbf{C}_{\text{fin}}$ also $A \in \mathbf{C}_{\text{fin}}$ such that (1) is a pullback in $\mathbf{C}_{\text{fin}}$ along $\mathcal{M}_{\text{fin}}$.*
I_{fin} preserves pushouts (pullbacks) along $\mathcal{M}_{\text{fin}}$ *if each pushout (pullback) (1) in $\mathbf{C}_{\text{fin}}$ with $f \in \mathcal{M}_{\text{fin}}$ ($g \in \mathcal{M}_{\text{fin}}$) is also a pushout (pullback) in $\mathbf{C}$ with $f \in \mathcal{M}$ ($g \in \mathcal{M}$).*

Proposition 7 (Creation and Preservation of Pushout and Pullback). *Given an $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ the inclusion functor $I_{\text{fin}}\colon \mathbf{C}_{\text{fin}} \to \mathbf{C}$ creates pushouts and pullbacks along $\mathcal{M}$ and preserves pushouts and pullbacks along $\mathcal{M}_{\text{fin}}$.*

Proof (Idea). 1. I_{fin} creates pullbacks along $\mathcal{M}$, because $\mathcal{M}$ is closed under pullbacks and composition and, therefore, each $\mathcal{M}$-subobject of A in Definition 8 is also an $\mathcal{M}$-subobject of B. Hence, B being finite implies that A is finite.
2. I_{fin} creates pushouts along $\mathcal{M}$, because $\mathcal{M}$ is closed under pushouts and, moreover, we can show (using the weak VK property) that each $\mathcal{M}$-subobject of D in Definition 8 corresponds up to isomorphism to a pair of $\mathcal{M}$-subobjects of B and C obtained by pullback constructions. Hence, B and C being finite implies that D is finite.
3. I_{fin} preserves pushouts along $\mathcal{M}_{\text{fin}}$, because given pushout (1) in $\mathbf{C}_{\text{fin}}$ with $f \in \mathcal{M}_{\text{fin}}$ also $f \in \mathcal{M}$. Since I_{fin} creates pushouts along $\mathcal{M}$ by Item 2, the pushout $(1')$ of $f \in \mathcal{M}$ and h in $\mathbf{C}$ is also a pushout in $\mathbf{C}_{\text{fin}}$. By uniqueness of pushouts this means that (1) and $(1')$ are isomorphic and hence (1) is also a pushout in $\mathbf{C}$.
4. Similarly, we can show that I_{fin} preserves pullbacks along $\mathcal{M}_{\text{fin}}$ using the fact that I_{fin} creates pullbacks along $\mathcal{M}$ as shown in Item 1. □

Now we are able to show the second main result.

Theorem 2. *The finitary restriction* $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ *of any* $\mathcal{M}$*-adhesive category* $(\mathbf{C}, \mathcal{M})$ *is a finitary* $\mathcal{M}$*-adhesive category.*

Proof. According to Remark 4, an object A in $\mathbf{C}$ is finite in $(\mathbf{C}, \mathcal{M})$ if and only if it is finite in $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$. Hence, all objects in $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ are finite.
Moreover, $\mathcal{M}_{\text{fin}}$ is closed under isomorphisms, composition, and decomposition, because this is valid for $\mathcal{M}$. $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ has pushouts along $\mathcal{M}_{\text{fin}}$ because $(\mathbf{C}, \mathcal{M})$ has pushouts along $\mathcal{M}$ and I_{fin} creates pushouts along $\mathcal{M}$ by Proposition 7. This implies also that $\mathcal{M}_{\text{fin}}$ is preserved by pushouts along $\mathcal{M}_{\text{fin}}$ in $\mathbf{C}_{\text{fin}}$. Similarly, $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ has pullbacks along $\mathcal{M}_{\text{fin}}$ and $\mathcal{M}_{\text{fin}}$ is preserved by pullbacks along $\mathcal{M}_{\text{fin}}$ in $\mathbf{C}_{\text{fin}}$.

Finally, the weak VK property of $(\mathbf{C}, \mathcal{M})$ implies that of $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ using that I_{fin} preserves pushouts and pullbacks along $\mathcal{M}_{\text{fin}}$ and creates pushouts and pullbacks along $\mathcal{M}$. □

A direct consequence of Theorem 2 is the fact that finitary restrictions of $(\mathbf{Sets}, \mathcal{M})$, $(\mathbf{Graphs}, \mathcal{M})$, $(\mathbf{Graphs_{TG}}, \mathcal{M})$, $(\mathbf{ElemNets}, \mathcal{M})$, $(\mathbf{PTNets}, \mathcal{M})$, and $(\mathbf{AGraphs_{ATG}}, \mathcal{M})$ are all finitary $\mathcal{M}$-adhesive categories satisfying not only the axioms of $\mathcal{M}$-adhesive categories, but also the additional HLR-requirements stated in Theorem 1, where, however, the existence of finite coproducts in $(\mathbf{AGraphs_{ATG}}, \mathcal{M})$ is valid (see [2]), but cannot be concluded from the existence of $\mathcal{M}$-initial objects as required in Items 3 and 4 of Theorem 1. Note, that I is an $\mathcal{M}_{\text{fin}}$-initial object in $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ if I is an $\mathcal{M}$-initial object in $(\mathbf{C}, \mathcal{M})$.

Remark 5. From Theorem 1 and Theorem 2 we can conclude that the main results for the DPO approach, like the Local Church-Rosser, Parallelism, Concurrency, Embedding, Extension, and Local Confluence Theorems, are valid in all finitary restrictions of $\mathcal{M}$-adhesive categories. This includes also corresponding results with nested application conditions [8], because shifts along morphisms and rules preserve finiteness of the objects occuring in the application conditions.

5 Functorial Constructions of Finitary $\mathcal{M}$-Adhesive Categories

Similar to general $\mathcal{M}$-adhesive categories, also finitary $\mathcal{M}$-adhesive categories are closed under product, slice, coslice, functor, and comma categories under suitable conditions [2]. It suffices to show this for functor and comma categories, because all others are special cases.

Proposition 8 (Finitary Functor Categories). *Given a finitary $\mathcal{M}$-adhesive category $(\mathbf{C}, \mathcal{M})$ and a category $\mathbf{X}$ with a finite class of objects then also the functor category $(\mathbf{Funct}(\mathbf{X}, \mathbf{C}), \mathcal{M}_F)$ is a finitary $\mathcal{M}$-adhesive category, where $\mathcal{M}_F$ is the class of all $\mathcal{M}$-functor transformations $t\colon F' \to F$, i. e., $t(X)\colon F'(X) \to F(X) \in \mathcal{M}$ for all objects X in $\mathbf{X}$.*

Proof (Idea). Functor categories over $\mathcal{M}$-adhesive categories are $\mathcal{M}$-adhesive due to Theorem 4.15.3 in [2]. It remains to show that each $F\colon \mathbf{X} \to \mathbf{C}$ is finite. Since $\mathbf{X}$ has a finite class of objects and for each X in $\mathbf{X}$ we have only finitely many $t(X)\colon F'(X) \to F(X) \in \mathcal{M}$, we also have only finitely many $t\colon F' \to F$ with all morphisms in $\mathcal{M}$ up to isomorphism. □

Remark 6. For infinite (discrete) $\mathbf{X}$ we have $\mathbf{Funct}(\mathbf{X}, \mathbf{C}) \cong \prod_{i\in\mathbb{N}} \mathbf{C}$. With $\mathbf{C} = \mathbf{Sets}_{\mathrm{fin}}$ the object $(2_i)_{i\in\mathbb{N}}$ with $2_i = \{1, 2\}$ has an infinite number of subobjects $(1_i)_{i\in\mathbb{N}}$ of $(2_i)_{i\in\mathbb{N}}$ with $1_i = \{1\}$, because in each component $i \in \mathbb{N}$ we have two choices of injective functions $f_{1/2} : \{1\} \to \{1, 2\}$. Hence $\mathbf{Funct}(\mathbf{X}, \mathbf{C})$ is not finitary, because $(2_i)_{i\in\mathbb{N}}$ in $\prod_{i\in\mathbb{N}} \mathbf{C}$ is not finite.

Proposition 9 (Finitary Comma Categories). *Given finitary $\mathcal{M}$-adhesive categories $(\mathbf{A}, \mathcal{M}_1)$ and $(\mathbf{B}, \mathcal{M}_2)$ and functors $F\colon \mathbf{A} \to \mathbf{C}$ and $G\colon \mathbf{B} \to \mathbf{C}$, where F preserves pushouts along $\mathcal{M}_1$ and G preserves pullbacks along $\mathcal{M}_2$, then the comma category $\mathbf{ComCat}(F, G; \mathcal{I})$ with $\mathcal{M} = (\mathcal{M}_1 \times \mathcal{M}_2) \cap \mathbf{ComCat}(F, G; \mathcal{I})$ is a finitary $\mathcal{M}$-adhesive category.*

Proof (Idea). Comma categories under $\mathcal{M}$-adhesive categories are $\mathcal{M}$-adhesive due to Theorem 4.15.4 in [2]. It remains to show that each (A, B, op) in $\mathbf{ComCat}$ $(F, G; \mathcal{I})$ is finite. Since $(\mathbf{A}, \mathcal{M}_1)$ and $(\mathbf{B}, \mathcal{M}_2)$ are finitary we have only finitely many choices for $m\colon A' \to A$ and $n\colon B' \to B$ in a subobject (A', B', op'). Moreover, we have at most one choice for each op'_k with $k \in \mathcal{I}$, since $G(n)$ is a monomorphism by G preserving pullbacks along $\mathcal{M}_2$. □

Remark 7. Note that $\mathcal{I}$ in $\mathbf{ComCat}(F, G; \mathcal{I})$ is not required to be finite.

6 Extension to Non-$\mathcal{M}$-Adhesive Categories

There are some relevant categories in computer science which are not $\mathcal{M}$-adhesive for non-trivial choices of $\mathcal{M}$. The categories **SGraphs** of simple graphs (i. e., there is at most one edge between each pair of vertices) and **RDFGraphs** of Resource Description Framework graphs [9, 10] are, e. g., only $\mathcal{M}$-adhesive if $\mathcal{M}$

is chosen to be bijective on edges which is not satisfactory, since $\mathcal{M}$ is the class used for transformation rules and these should be able to add and delete edges. This difference between multi and simple graphs is due to the fact that colimits implicitly identify equivalent edges for simple graphs and similar structures and, hence, behave radically differently than in the case of multi graphs.

Similar behaviour can be expected for a wide variety of categories in which the objects contain some kind of relational structure. Since relational structures are omnipresent in computer science – in databases, non-deterministic automata, logical structures – the study of transformations for these categories is also highly relevant.

Moreover, pushout complements are not even unique in these categories leading to double-pushout transformations being non-deterministic even for determined rule and match. We can, however, canonically choose a minimal pushout complement (MPOC), which is the approach taken in [9, 10]. This leads to a new variant of the double-pushout transformation framework which is applicable to such categories of relational structures.

Therefore, it is interesting to explore to what extent the results on finitary $\mathcal{M}$-adhesive categories presented in this paper are also valid in such non-$\mathcal{M}$-adhesive categories in order to transfer as much as possible of the extensive theoretical results from $\mathcal{M}$-adhesive categories to the MPOC framework and possibly also other approaches.

Definition 9 ($\mathcal{M}$-Category). *A category* $\mathbf{C}$ *together with a class* $\mathcal{M}$ *of monomorphisms is called* $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$ *if* $\mathcal{M}$ *is closed under composition, decomposition and isomorphisms, pushouts and pullbacks along* $\mathcal{M}$ *exist, and* $\mathcal{M}$ *is closed under pushouts and pullbacks.*

An object A *in* $(\mathbf{C}, \mathcal{M})$ *is called* finite *if the number of* $\mathcal{M}$*-subobjects of* A *is finite and the* $\mathcal{M}$*-category is called* finitary *if each object* A *in* $(\mathbf{C}, \mathcal{M})$ *is finite.*

Propositions 1–5 regarding coproducts and factorizations are already valid for (finitary) $\mathcal{M}$-categories, where we need in addition $\mathcal{M}$-initial objects for Propositions 2 and 5. Moreover, Propositions 8 and 9 remain valid for finitary $\mathcal{M}$-categories, but this problem is open for the creation of pushouts in Proposition 7 and, hence, also for Theorem 2.

By contrast, initial pushouts as they are defined in Definition 6 and constructed in Proposition 6 do not, in general, exist in finitary $\mathcal{M}$-categories. The problem is that the squares between the initial pushout and the comparison pushouts have to be pushouts themselves. Therefore, we define a weaker variant of initial pushouts, which does not require these squares to be pushouts but just to be commutative.

Definition 10 (Weak Initial Pushout). *Given an* $\mathcal{M}$*-category* $(\mathbf{C}, \mathcal{M})$*, a pushout* (1) *as in Definition 6 over a morphism* $m\colon L \to G$ *with* $b, c \in \mathcal{M}$ *is called* weak initial *if for all pushouts* (2) *over* m *with* $b', c' \in \mathcal{M}$ *there exist unique morphisms* $b^*, c^* \in \mathcal{M}$*, such that* $b' \circ b^* = b$*,* $c' \circ c^* = c$*, and* (3) *commutes.*

Remark 8. Observe that in $\mathcal{M}$-adhesive categories each weak initial pushout is already an initial pushout, since the initial pushout can be decomposed by

$\mathcal{M}$-pushout-pullback-decomposition which holds in $\mathcal{M}$-adhesive categories, because the comparison pushout is also a pullback in $\mathcal{M}$-adhesive, but not in general $\mathcal{M}$-categories.

Now, we show the existence and construction of weak initial pushouts for finitary $\mathcal{M}$-categories, provided that $\mathcal{M}$-pushouts are closed under pullbacks in the following sense.

$$
\begin{tikzcd}
 & & B_1 \arrow[dr, "b_1"] \arrow[dd, "a_1"] & \\
B_{12} \arrow[urr, "u_1"] \arrow[rrr, "b"] \arrow[dr, "u_2"] \arrow[dd, "a_{12}"'] & & & L \arrow[dd, "m"] \\
 & B_2 \arrow[urr, "b_2"] \arrow[dd, "a_2"] & C_1 \arrow[dr, "c_1"] & \\
C_{12} \arrow[urr, "v_1"] \arrow[rrr, "c"] \arrow[dr, "v_2"] & & & G \\
 & C_2 \arrow[urr, "c_2"] & &
\end{tikzcd}
$$

Definition 11 (Closure of $\mathcal{M}$-Pushouts under Pullbacks). *Given an $\mathcal{M}$-category $(\mathbf{C}, \mathcal{M})$, we say that $\mathcal{M}$-pushouts are closed under pullbacks if for each morphism $m\colon L \to G$ and commutative diagram with pushouts over m in the right squares, pullbacks in the top and bottom and $b_1, b_2 \in \mathcal{M}$ (and, hence, $c_1, c_2, u_1, u_2, v_1, v_2 \in \mathcal{M}$) it follows that the diagonal square is a pushout.*

Proposition 10 (Existence of Weak Initial Pushouts). *Finitary $\mathcal{M}$-categories have weak initial pushouts, provided that $\mathcal{M}$-pushouts are closed under pullbacks.*

Proof (Idea). Similarly to Proposition 6, we obtain the weak initial pushout by finite $\mathcal{M}$-intersections B and C constructed by iterated pullbacks. Now, the closure of $\mathcal{M}$-pushouts under pullbacks is used to directly show that the diagonal square is a pushout in each iteration. □

Note that the required closure of $\mathcal{M}$-pushouts under pullbacks already holds in $\mathcal{M}$-adhesive categories. Moreover, the closure holds in the categories **SGraphs** and **RDFGraphs**, allowing us to construct weak initial pushouts in these categories.

Remark 9. Similar to Remark 3, weak initial pushouts allow to define a gluing condition, which in this case is necessary and sufficient for the existence and uniqueness of minimal pushout complements (see [10]).

7 Conclusion

We have introduced finite objects in weak adhesive HLR categories [2], called $\mathcal{M}$-adhesive categories for simplicity in this paper. This leads to finitary $\mathcal{M}$-adhesive categories, like the category $\mathbf{Sets}_{\text{fin}}$ of finite sets and $\mathbf{Graphs}_{\text{fin}}$ of finite graphs with class $\mathcal{M}$ of all monomorphisms. In order to prove the main results like the Local Church-Rosser, Parallelism, Concurrency, Embedding, Extension, and Local Confluence Theorems not only the well-known properties of $\mathcal{M}$-adhesive categories are needed in [2], but also some additional HLR-requirements, especially initial pushouts, which are important to define the gluing condition and pushout complements, but often tedious to be constructed explicitly. In this paper, we have shown that for finitary $\mathcal{M}$-adhesive categories initial pushouts can be constructed by finite $\mathcal{M}$-intersections. Moreover, also the other additional HLR-requirements are valid in finitary $\mathcal{M}$-adhesive categories and, hence, the main results are valid for all $\mathcal{M}$-adhesive systems with finite objects, which are especially important for most of the application domains.

In order to construct finitary $\mathcal{M}$-adhesive categories we can either restrict $\mathcal{M}$-adhesive categories to all finite objects or apply suitable functor and comma category constructions (known already for general $\mathcal{M}$-adhesive categories [2]).

Finally, we have extended some of the results to non-$\mathcal{M}$-adhesive categories, like the category of simple graphs. Although adhesive categories [1] are special cases of $\mathcal{M}$-adhesive categories for the class $\mathcal{M}$ of all monomorphisms we have to be careful in specializing the results to finitary adhesive categories. While an object is finite in an $\mathcal{M}$-adhesive category $\mathbf{C}$ if and only if it is finite in the finitary restriction $\mathbf{C}_{\text{fin}}$ (with $\mathcal{M}_{\text{fin}} = \mathcal{M} \cap \mathbf{C}_{\text{fin}}$) this is only valid in adhesive categories if the inclusion functor $I \colon \mathbf{C}_{\text{fin}} \to \mathbf{C}$ preserves monomorphisms. It is to our knowledge an open problem for which kind of adhesive categories this condition is valid. Concerning categories $\mathbf{C}$ with a class $\mathcal{M}$ of monomorphisms, called $\mathcal{M}$-categories, it is open whether there are non-$\mathcal{M}$-adhesive categories such that the finitary restriction $(\mathbf{C}_{\text{fin}}, \mathcal{M}_{\text{fin}})$ is a finitary $\mathcal{M}$-adhesive category. For non-$\mathcal{M}$-adhesive categories it would be interesting to find a variant of the Van-Kampen-property which is still valid and allows to prove at least weak versions of the main results known for $\mathcal{M}$-adhesive systems. The closure of $\mathcal{M}$-pushouts under pullbacks is a first step in this direction, because it allows to construct weak initial pushouts for finitary $\mathcal{M}$-categories.

It remains open to compare our notion of finite objects in $\mathcal{M}$-categories with similar notions in category theory [11, 7] and to investigate other examples of $\mathcal{M}$-categories. Moreover, the relationships to work on (finite) subobject lattices in adhesive categories in [12, 13] are a valuable line of further research.

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