Bigraphs meet Double Pushouts

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Contents

1 Preface

The intention of this contribution is to discuss the relationship between ”Bigraphical reactive systems” [Mil01] and the Double Pushout Approach for Graph Transformation Systens” [Ehr79, Roz97] on a conceptual level. For this purpose we give a short introduction to the main concepts of both approaches, especially to bigraphs and double pushout transformations. The relationship between both approaches has been established concerning the following aspects:

- Presentation and Composition of Graphs
- Categorical Frameworks and Transfer of Concepts
- Rewrite Relations an Transformations.

Especially we point out which concepts correspond to each other and which of them have no counterpart in the other one. In some cases we are able to provide missing counterparts. Concerning the presentation of both approaches we abstract from some details, which are not essential for the comparison within the scope of this paper. On the other hand we hope that our presentation is detailed enough for the bigraph and the double pushout community to achieve at least an intuitive understanding of each other. This should allow to present a more formal relationship between both approaches in forthcoming papers. In this sense we are confident that bigraphs and double pushouts are on the way to meet each other.

Acknowledgements

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2 Bigraphs versus Double Pushouts

The notion of bigraphs has been proposed by Milner [Mil01] at CONCUR 2001 as the basis for a model of mobile interaction based on joint work with Leifer [LM00] at CONCUR 2001 and several other papers on action calculi in the last decade. A bigraph consists of a ”topograph” and a ”monograph” representing locality and connectivity of reactive systems respectively. Bigraphs are equipped with reaction rules to form bigraphical reactive systems, which include versions of the $\pi - calculus$ and the ambient calculus. A more abstract categorical version of the bigraphical approach are "Reactive Systems", which allow to study reaction and transition relations with bisimilarity using variants of pushouts, called reactive and idem pushouts. On the other hand the double pushout approach, short DPO approach, for graph transformation has been proposed by Ehrig, Pfender and Schneider in [EPS73] as a general model for graph rewriting and transformation.

The first intention was to generalize Chomsky grammars and term rewriting systems from strings resp. trees to general graphs. A first introduction to this approach has been given at the "First International Workshop on Graph Grammar with Applications in Computer Science and Biology" (see [Ehr79]). Meanwhile the DPO-approach has been studied and further developed by a growing community leading to a well-developed graphical modeling technique with concurrent semantics and various applications in Computer Science and related areas [Roz97, EEKR99, EKMR99]. This approach is called DPO-approach, because a direct transformation $G \Longrightarrow H$ via p from graph G to graph H via a production p consists of two pushouts (1) and (2) in the category **Graphs** of graphs and graph morphisms

$$\begin{tikzcd} L \arrow[d] \arrow[dr, phantom, "(1)"] & I \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & R \arrow[d] \\ G & C \arrow[l] \arrow[r] & H \end{tikzcd}$$

the upper row of the double pushout represents the production $p = (L \leftarrow I \rightarrow R)$, where I is the interface between the left hand side L and the right hand side R of the production. The graph C represents the context graph, such that the gluing of L with C via I (resp. R with C via I) leads to the graph G (resp. H) (see 5.2 for more details). This approach has been generalized from **Graphs** to other categories, where the objects can be considered as "high-level structures", leading to the concept of "high-level replacement systems" in [EHKP91].

3 Presentation and Composition of Graphs

In this section we discuss in more detail in which way graphs and composition of graphs are modeled in the bigraphical approach in contrast to the DPO approach. We show that graphs are presented as morphisms in the first and as objects in the second approach. Composition of graphs is modeled by composition of morphisms in the first approach in contrast to gluing constructions defined by pushouts in the second one. Finally we discuss the relationship between composition and gluing of graphs in both approaches.

3.1 Bigraphs in the Bigraphical Approach

Similar to the idea of Lawiere categories, where terms are presented as morphisms, also bigraphs are presented as morphisms between suitable interfaces. The essential idea of a bigraph is to comprise two different aspects of the sate of a reactive system. The locality is presented by a hierarchical structure on the set of nodes, while the connectivity structure is presented by connections between distinguished ports of the nodes. In order to be able to study these two different aspects separately, both structures are presented as separate graphs, called "topograph" an "monograph" respectively, which, however, share the same set of nodes. The hierarchical structure of a topograph is presented by a forest of trees. The connectivity structure of a monograph, on the other hand, is presented by an equivalence relation on the ports of the nodes and the interfaces, where each equivalence class of ports can be represented by a hyperedge connecting all the ports in this equivalent class. In the following we give a more detailed presentation of topographs, monographs and bigraphs respectively.

A topograph $G_T = (V, ctrl, prt) : m \to n$ consists of a set V of nodes, finite numbers m and n, called inner and outer width, a control function $ctrl : V \to K$ (where K is a given signature with arity $ar : K \to \mathbb{N}$), and an acyclic parent function $prt : m - V \to V + n$ (where $+$ is disjoint union and m, n are now considered as sets $m = 0, .., m-1$ and $n = 0, .., n-1$). In other words G^T is a forest of n trees with roots $0, .., n-1$ and m sites $0, .., m-1$. The sites are those leaves of the trees where other trees may be matched. This allows to define a composition $H^T \circ G^T : m \to l$ of topographs $G_T : m \to n$ and $H_T : n \to l$ leading to a precategory **TOP** of topographs. In fact the composition is only defined if the node sets of G^T and H^T are disjoint because this approach uses explicit names for nodes to handle sharing of nodes. This means that **TOP** becomes a precategory, where - in contrast to categories - composition of morphisms is defined only partially, Fortunately the categorical constructions needed in the bigraphical approach are well-defined already in precategories.

A monograph $G_M = (V, ctrl, \equiv) : X \to Y$ shares the same set of nodes V and control function $ctrl$ with G^T. Moreover X and Y are finite sets called conames and names of G^M, and $\equiv$ is an equivalence relation upon the set $X + P + Y$ of parts. Here the set P of inner ports in the disjoint union of node arities, i.e. $P = \sum_{v \in V} ar(ctrl(v))$. The composition $H^M \circ G^M : X \to Z$ of monographs $G_M : X \to Y$ and $H^M : Y \to Z$ is again only defined if the node sets of G_M and H_M are disjoint. The equivalence relation of $H^M \circ G^M$ is generated by the union of those from G_M and H^M, where elements from Y are deleted. this leads to a precategory **Mog** of monographs and - combining topo- and monographs - to a precategory **Big** of bigraphs $G = (V, ctrl, G_T, G_M) : I \to J$. Here $I =< m, X >$ and Here $J =< n, Y >$ are called inner and outer interfaces of G. This means that the bigraphical approach graphs (i.e. topographs, monographs and bigraphs) are morphisms in the corresponding precategories, whereas the interfaces (resp. width, names and conames) are objects.

3.2 Graphs in the DPO Approach

In the DPO framework for graph transformation we have - in contrast to the bigraphical framework - general graphs $G = (E, V, s, t)$ with sets E and V of edges resp. vertices and functions $s : E \to V, t : E \to V$, called source ant target respectively. More precisely the DPO framework has been formulated in [EPS73, Ehr79] for labeled graphs, where edges and vertices are labeled using label alphabets. Meanwhile the DPO approach has been extended to several other variants of graphs. For graphs $G_1 = (E_1, V_1, s_1, t_1)$ and $G_2 = (E_2, V_2, s_2, t_2)$ a graph morphism $f = (f_E, f_V : G_1 \to G_2$ consists of functions $f_E : E_1 \to E_2$ and $f_V : V_1 \to V_2$, s.t. $f_V \circ s_1 = s_2 \circ f_E$ and $f_V \circ t_1 = t_2 \circ f_E$. This means that graphs and graph morphisms can be considered as algebras and homomorphisms in the sense of algebraic specifications [EM85] leading to the category **Graphs**.

In contrast to the bigraphical framework graphs are objects in the DPO framework and graph morphisms are morphisms in **Graphs**. Interfaces are also graphs, i.e. edges are allowed in interfaces, and hence also objects. The gluing of two graphs G_1 and G_2 with interface I and embeddings $f_1 : I \to G_1$, $f_2 : I \to G_2$ given by graph morphisms is defined by the pushout object G with graph morphisms $g_1 : G_1 \to G$ and $g_2 : G_2 \to G$, s.t. we obtain the following pushout (PO) in the category **Graphs**:

$$\begin{tikzcd} I \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, phantom, "(PO)"] & G_1 \arrow[d, "g_1"] \\ G_2 \arrow[r, "g_1"] & G \end{tikzcd}$$

3.3 Composition versus Gluing of Graphs

The gluing of graphs discussed above corresponds roughly to the composition of graphs in the bigraphical approach. A difference, however, is the fact that the graph G obtained by gluing contains the interface I via the graph morphism $g_1 \circ f_1 = g_2 \circ f_2 : I \to G$, while the construction $G_2 \circ G_1$ of composition for topo-, mono- and bigraphs $G_1 : I_1 \to I_2$ and $G_2 : I_2 \to I_3$ deletes the middle interface I_2. In the case of topographs the deletion can be also inplicitely handled by gluing: We use as embeddings the parent function $prt_2 : I_2 \to (V_2 + I_3)$ and the inclusion $i : I_2 \to (I_1 + V_1 + I_2)$ for $G_i = (V_i, ctrl_i, prt_i) : I_i \to I_{i+1}$ $(i = 1, 2)$. Roughly spoken the pushout graph G of prt_2 and i corresponds to the topograph composition $G_2 \circ G_1 : I_1 \to I_3$ including the interfaces. A corresponding gluing construction for the composition of monographs, where also the middle interface is deleted, is desirable, but not known to us up to now.

4 Categorical Frameworks and Transfer of Concepts

In this section we compare the categorical constructions used in the bigraphical and the DPO approach for graph transformation respectively. As pointed out already in section 2 both of them have been generalized to abstract categorical frameworks, which, however, are only roughly sketched in this section. The main aim of this section is to show that the constructions of slice, coslice and cospan categories are useful for the transfer of concepts and hence for the comparison of the two abstract frameworks.

4.1 Categorical Frameworks for both Approaches

In the previous section we have seen already that we need precategories **Top**, **Mog**, **Big** in the bigraphical approach in contrast to the use of categories of graphs in the DPO-approach. The fact that we have a precategory instead of a category, however, is not at all essential. It is more essential that graphs are presented by morphisms in the bigraphical approach in contrast to the representation by objects in the DPO-approach. This different point of view remains valid in the corresponding abstract frameworks:

The abstract framework for the bigraphical approach will be called reactive approach in the following, because it is based on a categorical framework, called wide reactive systems in [Mil01]. It is based on a suitable precategory with relative pushouts which are used to define the transition relation in this framework see 5.4 below. In fact, pushouts are not required in the reactive approach, because in general they do not exist in the precategories of the bigraphical approach.

The abstract framework for the DPO graph transformation approach will be called DPO-approach in the following, because it is based on general categories with pushouts in order to construct DPO transformations of general objects see 5.2 below. This general DPO approach was introduced in [EHKP91], where the corresponding abstract transformation systems are called high-level replacement systems. For the comparative study in this paper it is not essential to give a more detailed presentation of these two abstract frameworks, except for the formulation of rewrite relations and transformations in each of these frameworks which will be presented in section 5.

In order to relate corresponding concepts in both abstract frameworks we review in the following some general constructions in category theory and discuss in which sense they are useful for the transfer of concepts.

4.2 Cospan Category and Transfer of Concepts

The category **Cospan (C)** over a category **C** has as objects the objects of **C** and a morphism $G : I \to J$ in **Cospan (C)** is a conspan $I \to G \leftarrow J$ of morphisms in **C**. The composition $H \circ G : I \to K$ of morphisms $G : I \to J$ and $H : J \to K$ in **Cospan (C)** - represented by cospans $I \to G \leftarrow J$ and $J \to H \leftarrow K$ in **C** respectively - is represented by the cospan $I \to H \circ J \leftarrow K$ constructed by a pushout in **C** in the following diagram:

$$\begin{tikzcd} I \arrow[r] & G \arrow[dr] & J \arrow[l] \arrow[r] \arrow[d, phantom, "(PO)"] & H \arrow[dl] & K \arrow[l] \\ & & (H \circ J) & & \end{tikzcd}$$

For a graph G with interfaces I and J the graph with interfaces can be considered as a cospan $I \to G \leftarrow J$ and hence as a morphism in **Cospan (Graphs)**. This idea has been used already by Gaducci and Heckel [GH99] in an inductive view of DPO graph transformations. Moreover it has been used by Sobocinsky [Sob02] as abstraction of graphs and bigraphs in the bigraphical approach. More precisely Sobocinsky has presented a general construction of relative pushouts in **Cospan (C)** provided that **C** is a suitable category with pushouts.

For our comparison of abstract approaches the following observation is essential:

If the objects in a category **C** represent our graphs of interest, like in the DPO approach, then in **Cospan (C)** the morphisms represent the corresponding graphs. like in the reactive approach. Moreover, the cospan category is useful for the transfer of concepts in the following sense:

We formulate a concept of the reactive approach in a category **Cospan (C)** over **C** and interpret this construction in the category **C**. As a result we obtain a corresponding concept in the DPO approach. This will be shown in 5.3 and 5.5 below.

4.3 Coslice and Slice Categories with Transfer of Concepts

The category **Coslice (C)** over a category **C** has as objects coslices, which are morphisms in **C** of the form $a : \varepsilon \to I$, short (I, a), where ε is a fixed object in **C**. A morphism $D : (I, a) \to (J, b)$ in **coslice (C)** is a morphism $D : I \to J$ in **C** with $Doa = b$.

Dually to **Coslice (C)** the category **Slice (C)** over **C** has as objects slices $a : I \to \varepsilon$ with fixed object ε, and morphisms in **Slice (C)** from $a : I \to \varepsilon$ to $b : J \to \varepsilon$ are morphisms $D : I \to J$ in **C** with $b \circ D = a$. Composition of morphisms in **Slice (C)** and **Coslice (C)** are defined by composition in **Slice (C)**.

The slice and the coslice construction and category have been considered by Cattani, Leifer and Milner already in [CLM00] in connection with the bigraphical approach. For our comparison of approaches the following is essential:

If the morphisms in a category **C** represent our graphs of interest, like in the reactive approach, then in **Slice (C)** and **Coslice (C)** the objects represent the corresponding graphs, like in the DPO approach.

Dually to the transfer of concepts from the reactive approach to the DPO- approach using the cospan category we might expect a transfer in the opposite direction using the slice of the coslice category. Unfortunately this seems to be difficult or not possible at all. But both categories are of interest for the transfer of concepts within the reactive approach.

In fact, the slice category allows to transfer coproducts and pushouts into slice sums and relative pushouts respectively as introduces in [CLM00]. In more detail a coproduct resp. pushout in **Slice (C)** corresponds exactly to a slice sum resp. realtive pushout in **C**. This corresponcende allows to conclude directly that slice sums resp. relative pushouts have the same nice properties (e.g. composition and decomposition) like coproducts and pushouts.

The coslice category on the other hand allows to transfer coproducts and slice sums into pushouts and relative pushouts respectively. This means that a coproduct resp. slice sum in **Coslice (C)** corresponds exactly to a pushout resp. relative pushout in

C. Hence in order to study existence and construction of pushouts or relative pushouts in a category **C** like **Top**, **Mog** or **Big** 3.1 it is equivalent to study coproducts or slice sums in **Coslice (C)** or any suitable category **C'** which is isomorphic or eqivalent to **Coslice (C)**. In [CLM00] a category **C** of action graphs, which is in some sense a predecessor of **Top**, **Mog** and **Big** has been studied concerning pushouts and relative pushouts. According to the transfer of concepts discussed above a suitable category **C'** of embeddings has been constructed and shown to be isomorphic to **Coslice (C)**. Now coproducts and slice sums have been studied in **C'** leading to pushouts and relative pushouts in **C** via the isomorphism **C'** $\cong$ **Coslice (C)**.

5 Rewrite Relations and Transformations

In this section we introduce the rewrite relations and transformations in both abstract approaches in more detail and compare them with each other. First we discuss the reaction relation and corresponding reaction steps in the reactive approach, where DPO transformations are the counterpart in the DPO approach. Then we introduce the main concept of the reactive approach, called transition relation, and a corresponding notion of transition steps. The essential idea of the transition relation is the possibility to borrow a context from the environment in order to be able to perform a transition step. Such a concept is not known in the DPO approach up to now, but a corresponding new concept is defined in this section as counterpart for transition steps in the reactive approach.

5.1 Reaction Relation and Step

In the reactive approach a rewrite rule is called reaction rule and consists essentially of a pair (r, r') of morphisms $r, r' : \varepsilon \to I$ in **A**, where ε is a fixed object in **A** and generalizing $(0, \emptyset)$ in **Big**. Morphisms $a : \varepsilon \to I$ in **A** are called agents and general morphisms $D : I \to J$ are called contexts in **A**. A reaction relation between agents $a, a' : \varepsilon \to J$ is defined, written

$$a \to a'$$

if there exists a reaction rule r, r' and a context D such that $a = D \circ r$ and $a' = D \circ r'$ in **A** are defined as shown in the following commutative diagram:

$$\begin{tikzcd}
E \arrow[r, "r"] \arrow[dr, "a"'] & I \arrow[d, "D"] \arrow[dl, phantom, "(=)" near start] \arrow[dr, phantom, "(=)" near start] & E \arrow[l, "r'"'] \arrow[dl, "a'"] \\
& I &
\end{tikzcd}$$

Note that the wide reactive approach only defines a relation $a \to a'$ between agents, not showing the rule (r, r') and the context D. Making both of them explicit we call

$a \to a'$ via (r, r') and D

A reaction step with rule (r, r') and context D, if we have $a = D \circ r$ and $a' = D \circ r'$ in $\mathbf{A}$ as above.

5.2 DPO Transformation

a rewrite rule in the DPO approach is a rule or production $p = (L \leftarrow I \rightarrow R)$ as discussed in section 2. A context for p in this framework is an object "C" together wit ha morphism $i : I \rightarrow C$. An application of rule p to this context leads to a (direct) transformation

$G \Longrightarrow H$ via p with context $i : I \rightarrow C$,

where G and H are constructed as pushouts (1) resp. (2) in the following diagram

$$\begin{tikzcd}
L \arrow[d, "m"'] \arrow[dr, phantom, "(1)"] & I \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & R \arrow[d, "n"] \\
G & C \arrow[l] \arrow[r] & H
\end{tikzcd}$$

In practical examples for the DPO-approach, however, we do not apply in general a rule p to a context C via i, but we apply p to an object G via a match morphism $m : L \rightarrow G$. The match morphism determines where L, the redex of the rule, occurs in G. In fact, there may be several different match morphisms or none at all. An essential step in order to apply p at m is now to find an object C and morphisms $i : I \rightarrow C$ and $C \rightarrow G$ such that (1) becomes a pushout complement construction. In the category **Graphs** - and similarly in other categories - the match m has to satisfy a suitable gluing condition in order to be able to construct the pushout complement C. Otherwise the rule r is not applicable at m. If $I \rightarrow L$ is injective, then the pushout complement construction is unique up to isomorphism. In the graph case this first step corresponds to removing from G all items of the graph L (more precisely $m(L)$) which are not inthe interface I (more precisely not in the image of I in G). The result is the context graph C sucht that the gluing of L and C via I leads to the graph G, i.e. (1) becomes a pushout in **Graphs**. In a second step we construct the gluing of R and C via I leading to the graph H via pushout (2).

5.3 Relationship between Reaction Steps and DPO Transformations

Applying a rule p to an object G via match $m : L \rightarrow G$ in the DPO approach, it corresponds in the reactive approach to the situation that we have given a reaction rule (r, r') and an agent $a : \varepsilon \rightarrow J$. In the first step we would have to check the existence and to give a construction of a context $D : I \rightarrow J$ such that $D \circ r = a$, and in the second step we can construct the agent a' by composition $a' = D \circ r'$.

$$\begin{tikzcd} E \arrow[r, "r"] \arrow[dr, "a"'] & I \arrow[d, "D"] \arrow[dl, phantom, "(1)"] \arrow[dr, phantom, "(2)"] & E \arrow[l, "r'"'] \arrow[dl, "a'"] \\ & I & \end{tikzcd}$$

Now let us show how we can use the cospan idea to construct from a reaction step in the wide reactive approach, given by triangles (1) and (2) above, a corresponding DPO transformation $a \Longrightarrow a'$ via (p, D), where p is constructed from the reaction rule (r, r'). For this purpose we assume that the triangles (1) and (2) are given in a cospan category **Cospan(C)** over **C**, i.e. we have cospans $\varepsilon \to r \leftarrow I$, $\varepsilon \to r' \leftarrow I$ and $I \to D \leftarrow J$ leading to cospans $\varepsilon \to a \leftarrow J$ and $\varepsilon \to a' \leftarrow J$ by composition in **Cospan(C)**. Since composition in **Cospan(C)** is defined via pushouts in **C**, we obtain the following diagram, where (3) and (4) is a double pushout in **C** leading to the DPO transformation $a \Longrightarrow a'$ via (p, D).

$$\begin{tikzcd} E \arrow[r] & r \arrow[d] \arrow[r, phantom, "(3)"] & I \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, "(4)"] & r' \arrow[d] & E \arrow[l] \\ & a & D \arrow[l] \arrow[r] & a' & \\ & & I \arrow[u] & & \end{tikzcd}$$

5.4 Transition Relation and Step

In order to be able to model all kinds of transitions in the $\pi - calculus$ and in the ambient calculus it is not sufficient to consider in the reactive approach reaction relations as introduced above. In fact, we need a more general rewrite relation, called transition relation in [Mil01]. The ???????? idea is that we may have to borrow a context F in order to be able to apply a reaction rule $(r : \varepsilon \to I, r' : \varepsilon \to I)$ to an agent $a : \varepsilon \to J$. The problem is how to construct an additional context F in some minimal way, such that the following diagram (1) commutes:

$$\begin{tikzcd} E \arrow[r, "a"] \arrow[d, "r"'] \arrow[dr, phantom, "(1)"] & J \arrow[d, "f"] \\ I \arrow[r, "D"'] & K \end{tikzcd}$$

The required minimality would be satisfied if (1) would be a pushout in the category **CAT**A. However - as discussed above - we cannot assume to have pushouts in **CAT**A.

For this reason the reactive approach requires a slightly weaker version of a pushout, called relative pushout (see 4.3). Roughly spoken a relative pushout is a pushout for the pair (r, a) with respect to a given upperbound. Diagram (1) is called idem pushout, if it is a relative pushout w.r.t. the apper bound (D, F). Fortunately the categories **Top**, **Mog** and **Big** have relative pushouts and hence idem pushouts such that they can be required in the framework of reactive systems.

The essential idea of the transition relation in [Mil01] is the following:

The triple (a, F, a') with agents $a : \varepsilon \to J$, $a' : \varepsilon \to K$ and context $F : J \to K$ is a transition, written

$$a \longrightarrow^{F} a',$$

if there exists a reaction rule (r, r') with agents $r : \varepsilon \to I, r' : \varepsilon \to I$ and a context $D : I \to K$ such that diagram (1) above is an idem pushout and $a' = D \circ r'$. Similar to 5.1 we will call in this case

$a \overset{F}{\Longrightarrow} a'$ via (r, r') and (D, F)

a transition step, where the rule and the contexts are shown explicitly.

$$\begin{tikzcd} E \arrow[r, "r"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & I \arrow[d, "D"] \arrow[r, phantom, "(2)"] & E \arrow[l, "r'"'] \arrow[dl, "a'"] \\ J \arrow[r, "f"'] & K & \end{tikzcd}$$

The main aim of the abstract framework of wide reactive systems in [Mil01] is to define wide bisimilarity for agents $a, b : \varepsilon \to J$ w.r.t. the transition relation sketched above and to show that wide bisimilarity is a congruence. This allows to conclude by instantiation a corresponding result in the bigraphical framework, which can be applied to suitable versions of the $\pi - calculus$ and the ambient calculus.

5.5 DPO-Transformations with Partial Match and Borrowed Context

In 5.3 we have shown that DPO-transformations are the counterpart of reaction steps. Up to now, however, there is no DPO- counterpart for transition steps discussed in 5.4. According to the idea of the transition relation we will construct such a counterpart, called DPO transfromation with partial match and borrowed context. In contrast to the DPO approach we assume now that we only have a partial match of the redex L of our rule $p = (L \leftarrow I \to R)$ in G. This means that we have a partial match morphism $m' : L \mathrel{-\!\!\circ\!\!\!\rightarrow} G$, represented by a span

$$L \leftarrow^{i} D \to^{m} G$$

of total morphisms, where D corresponds to the domain of the partial morphism m' and $i : D \to L$ to the inclusion of D into L. According to a well-known construction in **Graphs**, we assume to have in our general DPO-framework a boundary construction for $i : D \to L$, i.e. a minimal interface B with morphism $b : B \to D$ such that the pair (b, i) has a pushout complement F in diagram (1) below. In a second step we construct the pushout $\overline{G}$ in (2) with morphisms $\overline{m} : L \to \overline{G}$ and $g : G \to \overline{G}$, where $\overline{G}$ is the minimal extension of G such that we obtain a total match $\overline{m} : L \to \overline{G}$. In steps (3) and

(4) we construct a standard DPO transformation from $\overline{G}$ to H via p as discussed in 5.2. The following diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "b"'] \arrow[dr, phantom, "(1)"] & F \arrow[d] & & \\
D \arrow[r, "i"] \arrow[d, "m"'] \arrow[dr, phantom, "(2)"] & L \arrow[d, "\overline{m}"] \arrow[dr, phantom, "(3)"] & I \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "(4)"] & R \arrow[d] \\
G \arrow[r, "g"] & \overline{G} & C \arrow[l] \arrow[r] & H
\end{tikzcd}$$

consisting of 4 pushouts (1) - (4) is called DPO-transformation from G to H with rule p, partial match $m' = (i, m)$ and borrowed context F, written

$G \overset{F}{\Longrightarrow} H$ via (p, m').

Note that according to the composite pushout (1) + (2) $\overline{G}$ is the gluing of G and F via B. This means that F is the minimal context, which has to be borrowed and glued to G, such that the partial match m' can be extended to a total match $\overline{m} : L \to \overline{G}$.

The relationship between reaction steps and DPO-transformations in 5.3 can now be extended to a relationship between transition steps in 5.4 and DPO transformations with partial match and borrowed context. In fact, diagram (1) in 5.4 in **Cospan(C)** interpreted in **C** leads to the two pushout diagrams (1) + (2) and (3) above, while the triangle (2) in 5.4 leads to pushout (4) above. Moreover, the minimality of diagram (1) in 5.4 expressed by the idem pushout property corresponds to the fact that for given $i : D \to L$ the boundary B and the borrowed context F are minimal such that L is the gluing of D and F via B in (1) above.

6 Conclusion

In this contribution we have discussed the relationship between bigraphs, bigraphical and reactive systems on one hand and graphs, DPO graph transformation and DPO transformations in general categrries on the other hand. The correspondence beween the first and the second approach has been established using the cospan construction and the corresponding category **Cospan(C)** over (C). We have shown via this correspondence that a DPO transformation is the counterpart of a reaction step (relation) and the new concept of a DPO transformation with partial match and borrowed context is the counterpart of a transition step (relation) in the reactive systems approach. In addition to the cospan construction for the category **Cospan(C)** also the slice and the coslice constructions for the categories **Slice(C)** resp. **Coslice(C)** allows to define an interesting transfer of concepts. This has already been observed in [CLM00] but deserves a more detailed study in forthcoming papers. Moreover, it remains open to transfer the main bisimilarity result in the first approach into the DPO framework. vice versa it might be interesting to transfer constructions like parallel, concurrent and amalgamated rules and corresponding transformations from the DPO approach to reactive systems.

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