Adhesive High-Level Replacement Categories and Systems

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Abstract. Adhesive high-level replacement (HLR) categories and systems are introduced as a new categorical framework for graph transformation in a broad sense, which combines the well-known concept of HLR systems with the new concept of adhesive categories introduced by Lack and Sobociński.
In this paper we show that most of the HLR properties, which had been introduced ad hoc to generalize some basic results from the category of graphs to high-level structures, are valid already in adhesive HLR categories. As a main new result in a categorical framework we show the Critical Pair Lemma for local confluence of transformations. Moreover we present a new version of embeddings and extensions for transformations in our framework of adhesive HLR systems.

1 Introduction

High-level replacement systems have been introduced in [1] to generalize the well-known double pushout approach from graphs [2] to various kinds of high-level structures, including also algebraic specifications and Petri nets. In order to generalize basic results, like the local Church-Rosser, parallelism and concurrency theorem, several different conditions have been introduced in [1], called HLR conditions. The theory of HLR systems has been applied to a large number of example categories, where all the HLR conditions have been verified explicitly. Unfortunately, however, these conditions have some kind of ad hoc character, because they are just a collection of all the properties which are used in the categorical proofs of the basic results. Up to now it has not been analyzed how far these HLR properties are independent from each other or are consequences of a more general principle.

This problem concerning the ad hoc character of the HLR conditions has been solved recently by Lack and Sobociński in [3] by introducing the notion of adhesive categories. They have shown that the concept of "van Kampen squares", short VK squares, known from topology [4], can be considered as such a general principle. Roughly spoken a VK square is a pushout square which is stable under pullbacks. The key idea of adhesive categories is the requirement that pushouts along monomorphisms are VK squares. This property is valid not

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only in the categories **Sets** and **Graph**, but also in several varieties of graphs, which have been used in the theory of graph grammars and graph transformation [5] up to now. On the other hand Lack and Sobociński were able to show in [3] that most of the ad hoc HLR conditions required in [1] can be shown for adhesive categories. Together with the results in [1] this implies that the basic results for the theory of graph transformation mentioned above are valid in adhesive categories, where only the Parallelism Theorem requires in addition the existence of binary coproducts.

Unfortunately the concept of adhesive categories incorporates an important restriction, which rules out several interesting application categories. The HLR framework in [1] is based on a distinguished class M of morphisms, which is restricted to the class of all monomorphisms in adhesive categories. This restriction rules out the category (**SPEC**, M) of all algebraic specifications with class M of all strict injective specification morphisms (see [1]) and several other integrated specification techniques like algebraic high-level nets [6, 7] and different kinds of attributed graphs [8, 9], which are important in the area of graph transformation and HLR systems.

In this paper we combine the advantages of HLR and of adhesive categories by introducing the new concept of "adhesive HLR categories". Roughly spoken an adhesive HLR category is an adhesive category with a suitable subclass M of monomorphisms, which is closed under pushouts and pullbacks. As main results of this paper we are able to show that adhesive HLR categories are closed under product, slice, coslice and functor category constructions and that most of the important HLR properties of [1] are valid. These results are generalizations of corresponding results in [3], where we remove the restrictions, that M is the class of all monomorphisms and that adhesive categories in [3] are required to have all pullbacks instead of pullbacks along M-morphisms only.

In sections 2 - 4 of this paper we review and recover the basic results for HLR systems in [1] and adhesive grammars in [3] in the framework of adhesive HLR categories and systems. Moreover, we present in section 5 a new version of the results for embedding and extension of transformations [2, 10, 11]. This is the basis to show in section 6 another main result of this paper: For the first time we present a categorical version of the Critical Pair Lemma for local confluence of transformations, discussed for hypergraphs in [12] and attributed graphs in [9], in our new framework of adhesive HLR systems.

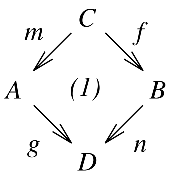
For lack of space we only give proof ideas for some of our results in this paper. For a more detailed version we refer to our technical report [13].

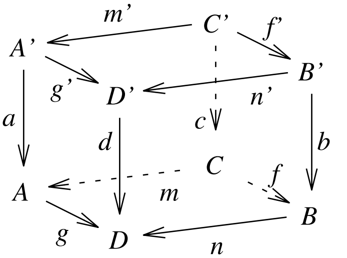
2 Review of Van Kampen Squares and Adhesive Categories

In this section we review adhesive categories as introduced by Lack and Sobiciński in [3].

The basic notion of adhesive categories is that of a so called van Kampen square. The intuitive idea of a van Kampen square is that of a pushout which is stable under pullbacks and vice versa pushout preservation implies pullback stability. The name van Kampen derives from the relationship between these squares and the Van Kampen Theorem in topology [4].

Definition 1 (van Kampen square). *A pushout (1) is a van Kampen (VK) square, if for any commutative cube (2) with (1) in the bottom and back faces being pullbacks holds: the top is pushout ⇔ the front faces are pullbacks.*





(2)

In the definition of adhesive categories only those VK squares are considered, where m is a monomorphism. In this case the square is called a pushout along a monomorphism. The first interesting property of VK squares in [3] shows that in this case also n is a monomorphism and the square is also a pullback.

Definition 2 (adhesive category). *A category* $\boldsymbol{C}$ *is an adhesive category, if*

1. $\boldsymbol{C}$ *has pushouts along monomorphisms, i.e. pushouts, where at least one of the given morphisms is monomorphism,*
2. $\boldsymbol{C}$ *has pullbacks,*
3. *pushouts along monomorphisms are VK squares.*

The most basic example of an adhesive category is the category **Sets** of sets. Moreover it is shown in [3] that adhesive categories are closed under product, slice, coslice and functor category construction. This implies immediately that also the category **Graphs** of graphs $G = (E \overset{s,t}{\rightrightarrows} V)$, and also several variants like typed graphs, labelled graphs and hypergraphs are adhesive categories. This is a first indication that adhesive categories are suitable for graph transformation. Counterexamples for adhesive categories are **Pos** (partially ordered sets), **Top** (topological spaces), **Gpd** (groupoids) and **Cat** (categories), where pushouts along monomorphisms fail to be VK squares (see [3]).

The main reason why adhesive categories are important for the theory of graph transformation and its generalization to high-level replacement systems (see [1]) is the fact that most of the HLR conditions required in [1] are shown to be valid already in adhesive categories (see [3]). This implies that basic results like the Local Church-Rosser Theorem and the Concurrency Theorem (see [1]) are valid already in the framework of adhesive categories, while the Parallelism Theorem needs in addition the existence of binary coproducts.

The main advantage of adhesive categories compared with HLR categories in [1] is the fact that the requirements for adhesive categories are much more smooth than the variety of different HLR conditions in [1], which have been stated "ad hoc" as needed in the categorical proofs of the corresponding results mentioned above.

On the other hand HLR categories in [1] are based on a class M of morphisms, which is restricted to the class of all monomorphisms in adhesive categories. This rules out several interesting examples. In order to avoid this problem we combine the two concepts leading to the notion of adhesive HLR categories in the next section.

3 Adhesive HLR Categories

As motivated in the previous section we will combine the concepts of adhesive categories [3] and HLR categories [1] leading to the new concept of adhesive HLR categories in this section. Most of the results presented in this section are generalizations of results for adhesive categories in [3], but we present new interesting examples which are not instantiations of adhesive categories.

The main difference of adhesive HLR categories compared with adhesive categories is the fact that we consider a suitable subclass M of monomorphisms instead of the class of all monomorphisms. Moreover we require only pullbacks along M-morphisms and not for general morphisms.

Definition 3 (adhesive HLR category ($\boldsymbol{C}$, M)). *A category $\boldsymbol{C}$ with a morphism class M is called adhesive HLR category, if*

1. *M is a class of monomorphisms closed under isomorphisms and closed under composition ($f : A \to B \in M$, $g : B \to C \in M \Rightarrow g \circ f \in M$) and decomposition ($g \circ f \in M$, $g \in M \Rightarrow f \in M$),*
2. *$\boldsymbol{C}$ has pushouts and pullbacks along M-morphisms and M-morphisms are closed under pushouts and pullbacks,*
3. *pushouts in $\boldsymbol{C}$ along M-morphisms are VK squares.*

Remark 1. Most of the results in this paper can also be formulated under slightly weaker assumptions, where the existence of pullbacks is required only if both given morphisms are in M and pushouts along M-morphisms are required to be M-VK squares only, i.e. only for the case $f \in M$ or $a, b, d \in M$. This weaker version is called "weak adhesive HLR category". But presently we have no interesting example of this weak case that is not also an adhesive HLR category.

Example 1. 1. All examples of adhesive categories are adhesive HLR categories for the class M of all monomorphisms. As shown in [3] this includes the category **Sets** of sets, **Graphs** of graphs and several variants of graphs like typed, labelled and hypergraphs discussed above. Moreover this includes the category **PT-Net** of place transition nets considered in [1].

2. The category (**Spec**, M_1) of algebraic specifications with class M_1 of all monomorphisms is not adhesive, because pushouts along monomorphisms are not necessarily pullbacks. But (**Spec**, M_2) with class M_2 of all strict injective specification morphisms is an HLR2 category in the sense of [1] and also an adhesive HLR category.

For similar reasons the category **AHL-Net** of algebraic high-level nets (see [7]) has to be considered with strict injective specification morphisms concerning the specification part of the net morphism.

3. An important new example is the category ($\mathbf{AGraphs_{ATG}}$, M) of typed attributed graphs with type graph ATG and class M of all injective morphisms with isomorphisms on the data part. In our paper [14] we explicitly show that this is an adhesive HLR category satisfying all additional HLR properties considered later in this paper.

The first important result shows that adhesive HLR categories are closed under product, slice, coslice and functor category construction. This allows to construct new examples from given ones.

Theorem 1 (construction of adhesive HLR categories). *Adhesive HLR categories can be constructed as follows:*

- *If ($\boldsymbol{C}$, M_1) and ($\boldsymbol{D}$, M_2) are adhesive HLR, then ($\boldsymbol{C} \times \boldsymbol{D}$, $M_1 \times M_2$) is adhesive HLR.*
- *If ($\boldsymbol{C}$, M) is adhesive HLR, then so are the slice category ($\boldsymbol{C}\backslash C$, $M \cap \boldsymbol{C}\backslash C$) and the coslice category ($C\backslash\boldsymbol{C}$, $M \cap C\backslash\boldsymbol{C}$) for any object C in $\boldsymbol{C}$.*
- *If ($\boldsymbol{C}$, M) is adhesive HLR, then every functor category ([$\boldsymbol{X}$, $\boldsymbol{C}$], M-functor transformations) is adhesive HLR.*

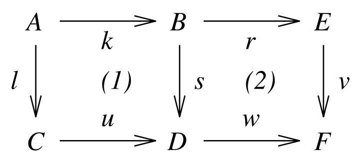
Remark 2. An M-functor transformation is a natural transformation $t : F \to G$ where all morphisms $t(X) : F(X) \to G(X)$ are in M.

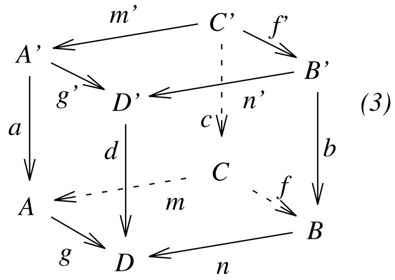
Proof idea. In the case of product and functor categories the properties of adhesive HLR categories can be shown componentwise. For slice and coslice categories some standard constructions for pushouts and pullbacks can be used to show the properties. □

The second important result shows that most of the HLR conditions stated in [1, 7] are already valid in adhesive HLR categories.

Theorem 2 (HLR properties of adhesive HLR categories). *Given an adhesive HLR category ($\boldsymbol{C}$, M), the following HLR conditions are satisfied.*

1. *Pushouts along M-morphisms are pullbacks.*
2. *Pushout-pullback decomposition: Given the following diagram with $l, w \in M$, (1) + (2) pushout and (2) pullback. Then (1) and (2) are pushouts and also pullbacks.*
3. *Cube pushout-pullback property: Given the following commutative cube (3), where all morphisms in top and bottom are in M, the top is pullback and the front faces are pushouts. Then we have: the bottom is pullback $\Leftrightarrow$ the back faces of the cube are pushouts.*





4. *Uniqueness of pushout complements for M-morphisms: Given $k : A \to B \in M$ and $s : B \to D$ then there is up to isomorphism at most one C with $l : A \to C$ and $u : C \to D$ such that diagram (1) is a pushout.*

Proof idea. These properties are shown for adhesive categories with class M of all monomorphisms in [3]. The proofs can be reformulated for any subclass M of monomorphisms as required for adhesive HLR categories. □

Remark 3. The HLR conditions stated above together with the existence of binary coproducts compatible with M (see Thm. 3) correspond roughly to the HLR conditions in [7] resp. HLR2 and two of the HLR2* conditions in [1]. The HLR2 condition of [1] stating that M is closed under isomorphisms is not needed in our context.

4 Adhesive HLR Systems

In this section we use the concept of adhesive HLR categories introduced in the previous sections to present the basic notions and results of adhesive HLR systems in analogy to HLR systems in [1]. The Local Church-Rosser Theorem and the Parallelism Theorem are shown to be valid in [1] for HLR1 categories, and the Concurrency Theorem for HLR2 categories, where the existence of binary coproducts is only needed for the Parallelism Theorem. Using the properties of adhesive HLR categories in the previous section we can immediatly conclude that the Local Church-Rosser Theorem and the Concurrency Theorem are valid in adhesive HLR categories and the Parallelism Theorem in adhesive HLR categories with binary coproducts.

Definition 4 (adhesive HLR system). *An adhesive HLR system $AS = (\boldsymbol{C}, M, S, P)$ consists of an adhesive HLR category ($\boldsymbol{C}$, M), a start object S and a set of productions P, where*

1. *a production $p = L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R$ consists of objects L, K and R called left-hand side, gluing object and right-hand side respectively, and morphisms $l : K \to L$, $r : K \to R$ with $l, r \in M$,*
2. *a direct transformation $G \overset{p,m}{\Longrightarrow} H$ via a production p and a morphism $m : L \to G$, called match, is given by the following diagram, called DPO-diagram, where (1) and (2) are pushouts,*

$$\begin{array}{ccccc} L & \xleftarrow{\;l\;} & K & \xrightarrow{\;r\;} & R \\ m\downarrow & (1) & \downarrow k & (2) & \downarrow n \\ G & \xleftarrow[\;f\;]{} & D & \xrightarrow[\;g\;]{} & H \end{array}$$

3. *a transformation is a sequence* $G_0 \Rightarrow G_1 \Rightarrow ... \Rightarrow G_n$ *of direct transformations, written* $G_0 \overset{*}{\Rightarrow} G_n$*,*
4. *the language* $L(AS)$ *consists of all objects* G *in* $\boldsymbol{C}$ *derivable from the start object* S *by a transformation, i.e.* $L(AS) = \{G \mid S \overset{*}{\Rightarrow} G\}$*.*

Remark 4. 1. An adhesive HLR system is on the one hand an HLR system in the sense of [1], where in [1] we have in addition a distinguished class T of terminal objects, and on the other hand an adhesive grammar in the sense of [3], provided that the class M is the class of all monomorphisms.
2. A direct transformation $G \overset{p,m}{\Longrightarrow} H$ is uniquely determined up to isomorphism by the production p and the match m, because due to Thm. 2 It. 4 pushout complements along M-morphisms in adhesive HLR categories are unique up to isomorphism.
3. All the examples for HLR1 and HLR2 systems considered in [1] and all systems over adhesive HLR categories considered in Ex. 1 are adhesive HLR systems, which includes especially the classical graph transformation approach in [2].

The following basic results are shown for HLR2 categories in [1] and they are rephrased for adhesive categories in [3]. According to Thm. 2 they are also valid for adhesive HLR systems. A more detailed version of these results is presented in [13].

Theorem 3 (Local Church-Rosser, Parallelism and Concurrency Theorem). *The Local Church-Rosser Theorems I and II, the Parallelism Theorem and the Concurrency Theorem as stated in [1] are valid for all adhesive HLR systems* $AS = (\boldsymbol{C}, M, S, P)$*. Only for the Parallelism Theorem we have to require in addition that* $(\boldsymbol{C}, M)$ *has binary coproducts which are compatible with* M*, i.e.* $m_1, m_2 \in M$ *implies* $m_1 + m_2 \in M$*.*

Proof. Follows from [1] and Thm. 2. □

5 Embedding and Extension of Adhesive HLR Transformations

In this section we present a categorical version of the Embedding Theorem for graph transformation (see [2]) using the concept of initial pushouts first introduced in [10]. The embedding theorem is not only important for the theory of graph transformation, but also for the component framework for system modelling introduced in [15]. In [11] it is shown how to verify the extension properties used in the generic component concept of [15] in the framework of HLR systems.

The Embedding Theorem and the Extension Theorem presented for adhesive HLR systems in this section combine the results for both areas and will also be used in the next section to show the Local Confluence Theorem. The key notion is the concept of initial pushouts, which formalizes the construction of boundary and context in [2]. The important new property going beyond [10] is the fact that initial pushouts are closed under double pushouts under certain conditions. As in Sec. 4 we assume also in this section that we have an adhesive HLR system.

We start with the definition of an extension diagram in the sense of [15, 11] which means that a transformation t is extended to a transformation t' via an extension morphism.

Definition 5 (extension diagram). *An extension diagram is a diagram (1)*

$$\begin{array}{ccc} G_0 & \overset{t}{\Longrightarrow}{}^{*} & G_n \\ k_0 \downarrow & (1) & \downarrow k_n \\ G_0' & \underset{t'}{\Longrightarrow}{}^{*} & G_n' \end{array}$$

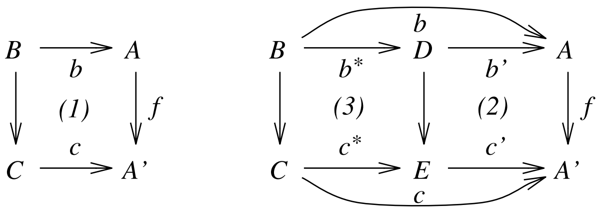
where $k_0 : G_0 \to G_0'$ is a morphism, called extension morphism, and $t : G_0 \overset{*}{\Rightarrow} G_n$ and $t' : G_0' \overset{*}{\Rightarrow} G_n'$ are transformations via the same productions $(p_0, ..., p_{n-1})$ and matches $(m_0, ..., m_{n-1})$ resp. $(k_0 \circ m_0, ..., k_{n-1} \circ m_{n-1})$ defined by the following DPOs.

$$\begin{array}{lccccc} p_i: & L_i & \longleftarrow & K_i & \longrightarrow & R_i \\ & m_i \downarrow & & \downarrow & & \downarrow \\ & G_i & \longleftarrow & D_i & \longrightarrow & G_{i+1} \\ & k_i \downarrow & & \downarrow & & \downarrow k_{i+1} \\ & G_i' & \longleftarrow & D_i' & \longrightarrow & G_{i+1}' \end{array} \qquad (i = 0, ..., n\text{-}1)$$

Remark 5. 1. The extension diagram (1) is completely determined (up to isomorphism) by $t : G_0 \overset{*}{\Rightarrow} G_n$ and $k_0 : G_0 \to G_0'$ (using the uniqueness of pushout complements).
2. Extension diagrams are closed under horizontal and vertical composition (using corresponding composition properties of pushouts).

The main problem is now to determine under which condition a transformation $t : G_0 \overset{*}{\Rightarrow} G_n$ and an extension morphism $k_0 : G_0 \to G_0'$ lead to an extension diagram. The key notion is that of an initial pushout, which will be required for the extension morphism k_0 in the consistency condition below.

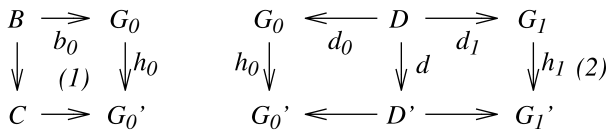
Definition 6 (initial pushout, boundary and context). *Given $f : A \to A'$ a morphism $b : B \to A$ with $b \in M$ is called boundary over f if there is a pushout complement such that (1) is an initial pushout over f. Initiality of (1) over f means, that for every pushout (2) with $b' \in M$ there exist unique morphisms $b^* : B \to D$ and $c^* : C \to E$ with b^*, $c^* \in M$ such that $b' \circ b^* = b$, $c' \circ c^* = c$ and (3) is pushout. Then B is called boundary object and C context w.r.t. $f : A \to A'$.*

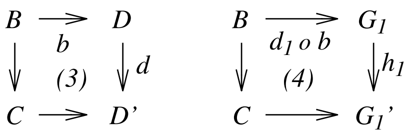


Remark 6. In the classical case of graph transformations [2] the boundary B of a graph morphism $f : A \to A'$ consists of all nodes in $b \in A$ such that $f(b)$ is adjacent to an edge in $A' \backslash f(A)$. These nodes are necessary to glue A to the context graph $C = A' \backslash f(A) \cup f(b(B))$ in order to obtain A' as gluing of A and C via B in the initial pushout (1).

As pointed out in the introduction of this section the closure of initial pushouts under double pushouts is an important technical lemma.

Lemma 1 (closure property of initial pushouts). *Let M' be a class of morphisms closed under pushouts and pullbacks along M-morphisms. Moreover we assume to have initial pushouts over M'-morphisms. Then initial pushouts over M'-morphisms are closed under double pushouts. That means given an initial pushout (1) over $h_0 \in M'$ and a double pushout diagram (2) with d_0, $d_1 \in M$, then (3) and (4) are initial pushouts over $d \in M'$ respectively $h_1 \in M'$ for the unique $b : B \to D$ with $d_0 \circ b = b_0$ obtained by initiality of (1).*



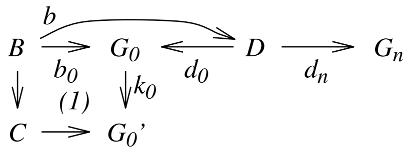


Proof idea. This can be shown shown stepwise for pushouts in the opposite and in the same direction by using the properties of M and M'. The complete proof can be found in [13]. □

The following consistency condition for a transformation $t : G_0 \overset{*}{\Rightarrow} G_n$ and an extension morphism $k_0 : G_0 \to G_0'$ means intuitively that the boundary B of k_0 is preserved by t. In order to formulate this property we use the notion of a derived span $der(t) = G_0 \leftarrow D \to G_n$ of the transformation t, which connects the first and the last object.

Definition 7 (derived span and consistency). *The derived span of a direct transformation $G \overset{p,n}{\Longrightarrow} H$ as shown in Def. 4 is the span $G \leftarrow D \to H$. The derived span $der(t) = (G_0 \overset{d_0}{\leftarrow} D \overset{d_n}{\to} G_n)$ of a transformation $t : G_0 \overset{*}{\Rightarrow} G_n$ is the*

composition via pullbacks of the spans of the corresponding direct transformations.
A morphism $k_0 : G_0 \to G'_0$ *is called consistent w.r.t. a transformation* $t : G_0 \overset{*}{\Rightarrow} G_n$ *with derived span* $der(t) = (G_0 \overset{d_0}{\leftarrow} D \overset{d_n}{\to} G_n)$ *if there exist an initial pushout (1) over* k_0 *and a morphism* $b \in M$ *with* $d_0 \circ b = b_0$.

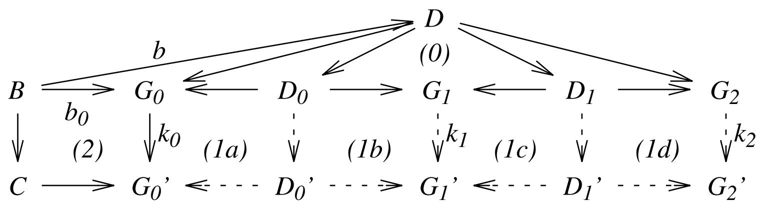


Remark 7. 1. The morphisms of the span $G \leftarrow D \to H$ are in M because M is closed under pushouts. This implies that the compositions of these spans exist and are M-morphisms, because pullbacks along M-morphisms exist and M-morphisms are closed under pullbacks in adhesive HLR categories.
2. The consistency condition in [2], called JOIN condition, requires a suitable family $b_i : B \to D_i$ of morphisms from the boundary B to the context graphs D_i of the direct transformations. In fact, our consistency condition is equivalent to the existence of a corresponding family $(b_i)_{i=0,\ldots,n-1}$.
3. For the definition of consistency and for Thm. 4 below – but not for Thm. 5 – it would be sufficient to require the existence of a pushout over k_0 instead of an initial one. Moreover we need only conditions 1 and 2 of Def. 3.

Now we are able to prove the Embedding and the Extension Theorem which show that consistency is sufficient and also necessary for the construction of extension diagrams. Moreover, we obtain a direct construction of the extension $k_n : G_n \to G'_n$ in the extension diagram.

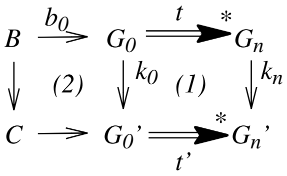
Theorem 4 (Embedding Theorem). *Given a transformation* $t : G_0 \overset{*}{\Rightarrow} G_n$ *and a morphism* $k_0 : G_0 \to G'_0$ *which is consistent w.r.t.* t*, then there is an extension diagram for* t *and* k_0 *(see (1) in Def. 5).*

Proof idea (n = 2). We construct pullback (0) leading to the derived span $G_0 \leftarrow D_0 \leftarrow D \to D_1 \to G_2$ of the transformation $t : G_0 \Rightarrow^* G_2$. Given k_0 consistent w.r.t. t we have initial pushout (2) over k_0 and $b : B \to D$.



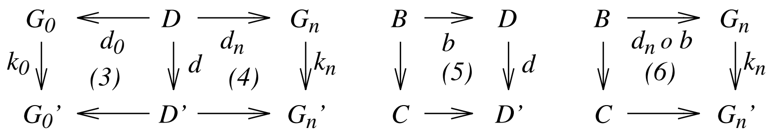
This leads to M-morphisms $B \to D_0$ and $B \to D_1$ such that first D'_0 can be constructed as pushout object of $B \to D_0$ and $B \to C$ leading by decomposition to pushout (1a) and by construction to pushout (1b). Then D'_1 can be constructed as pushout of $B \to D_1$ and $B \to C$ leading by decomposition to pushout (1c) and by construction to pushout (1d). The given transformation $t : G_0 \overset{*}{\Rightarrow} G_2$ together with the pushouts (1a) - (1d) constitutes the required extension diagram. □

Theorem 5 (Extension Theorem). *Given a transformation $t : G_0 \overset{*}{\Rightarrow} G_n$ with derived span $der(t) = (G_0 \overset{d_0}{\leftarrow} D \overset{d_n}{\rightarrow} G_n)$ and an extension diagram (1)*



with initial pushout (2) over $k_0 \in M'$ for some class M' closed under pushouts and pullbacks along M-morphisms and initial pushouts over M'-morphisms, then we have

1. *k_0 consistent w.r.t. $t : G_0 \overset{*}{\Rightarrow} G_n$ with morphism $b : B \rightarrow D$,*
2. *a direct transformation $G'_0 \Rightarrow G'_n$ via $der(t)$ and match k_0 given by pushouts (3) and (4) with d, $k_n \in M'$,*
3. *initial pushouts (5) and (6) over d resp. k_n.*



Remark 8. The extension theorem shows

1. Consistency of k_0 w.r.t. t is necessary for the existence of the extension diagram.
2. The extension diagram (1) can be represented by a direct transformation with match k_0 and comatch k_n.
3. The extension $k_n : G_n \rightarrow G'_n$ can be constructed by a pushout (6) of G_n and context C along the boundary B with $d_n \circ b : B \rightarrow G_n$.

Proof idea ($n = 2$). Given t and k_0 with initial pushout (2) and the extension diagram given by pushouts (1a) - (1d) in proof of Thm. 4, where D is pullback in (0). Initiality of (2) and pushout (1a) lead to $b_0^* : B \rightarrow D_0$ and by Lem. 1 to an initial pushout over k_1. This new initiality and pushout (1c) leads to $b_1 : B \rightarrow D_1$. The morphisms b_0^* and b_1 lead to an induced $b : B \rightarrow D$ - using the pullback properties of (0) - which allows to show consistency of k_0 w.r.t. t. This consistency immediately implies the pushout complement D' in (3) and pushout (4) of d and d_n. Finally, the double pushout (3), (4) implies by Lem. 1 initial pushouts (5) and (6) from (2). □

6 Critical Pairs and Local Confluence of Adhesive HLR Systems

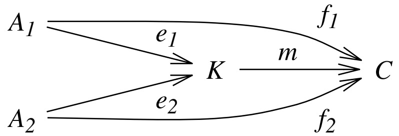
Critical pairs and local confluence have been studied for hypergraph transformations in [12] and for typed attributed graph transformation in [9]. In this section we present a categorical version in adhesive HLR categories. As additional requirements we only need an E'-M' pair factorization for cospans of morphisms - in analogy to the well-known epi-mono-factorization of morphisms - and initial pushouts over M'-morphisms. These assumptions are stated where necessary. Otherwise we only assume to have an adhesive HLR system.

It is well-known that local confluence and termination imply confluence. But we only analyze local confluence in this paper and no termination nor general confluence.

Definition 8 (confluence, local confluence). *A pair of transformations $H_1 \overset{*}{\Leftarrow} G \overset{*}{\Rightarrow} H_2$ is confluent if there are transformations $H_1 \overset{*}{\Rightarrow} X$ and $H_2 \overset{*}{\Rightarrow} X$. An adhesive HLR system is locally confluent, if this property holds for each pair of direct transformations, it is confluent, if it holds for all pairs of transformations.*

In order to define and construct critical pairs we introduce the notion of E'-M' pair factorization.

Definition 9 (E'-M' pair factorization). *An adhesive HLR category has E'-M' pair factorization, if M' is a class of morphisms closed under pushouts and pullbacks along M-morphisms and E' a class of morphism pairs with same codomain and we have for each pair of morphisms $f_1 : A_1 \to C$, $f_2 : A_2 \to C$ that there is an object K and morphisms $e_1 : A_1 \to K$, $e_2 : A_2 \to K$, $m : K \to C$ with $(e_1, e_2) \in E'$, $m \in M'$ such that $m \circ e_1 = f_1$ and $m \circ e_2 = f_2$.*



Remark 9. It is sufficient to require this property for matches $f_i = m_i : L_i \to G$ (i = 1, 2). The closure properties of M' are needed in Lem. 2 and Thm. 6.

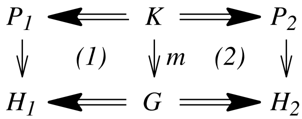
The intuitive idea of morphism pairs $(e_1, e_2) \in E'$ in most example categories is that the pair is jointly surjective resp. jointly epimorphic. This can be achieved in categories $\boldsymbol{C}$ with binary coproducts and E_0-M_0 factorization of morphisms, where $E_0 \subseteq Epis$ and $M_0 \subseteq Monos$. Given $A_1 \overset{f_1}{\to} C \overset{f_2}{\leftarrow} A_2$ we simply take an E_0-M_0 factorization $f = m \circ e$ of the induced morphism $f : A_1 + A_2 \to C$ and define $e_1 = e \circ i_1$ and $e_2 = e \circ i_2$, where i_1, i_2 are the coproduct injections. If the category has no binary coproducts, or the construction above is not always adequate - as in the case of typed attributed graph transformation - we may have another alternative to obtain an E'-M' pair factorization. In [14] an explicit E'-M' pair factorization for typed attributed graph transformation is provided, where M'-morphisms are not necessarily injective on the data type part and hence $M' \not\subseteq M$.

The main idea to prove local confluence is to show local confluence explicitely only for critical pairs based on the notion of parallel independence (see [1]).

Definition 10 (critical pair). *Given an E'-M' pair factorization, a critical pair is a pair of non-parallel independent direct transformations $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ such that $(o_1, o_2) \in E'$ for the corresponding matches o_1 and o_2.*

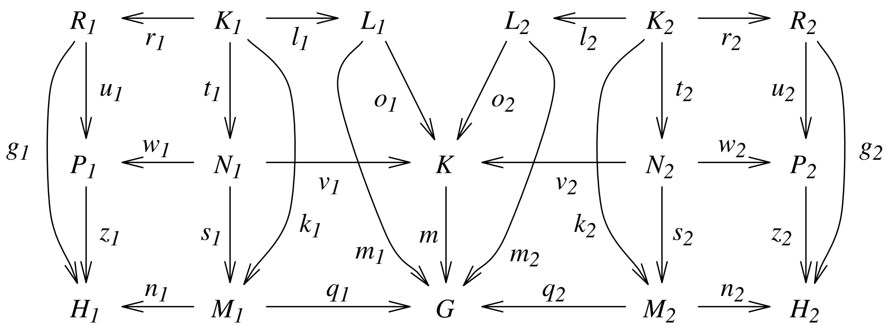
The first step towards local confluence is to show completeness of critical pairs.

Lemma 2 (completeness of critical pairs). *Consider an adhesive HLR system with E'-M' pair factorization and $M' \subseteq M$. For each pair of non-parallel independent direct transformations $H_1 \overset{p_1,m_1}{\Longleftarrow} G \overset{p_2,m_2}{\Longrightarrow} H_2$ there is a critical pair $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ with extension diagrams (1) and (2) and $m \in M'$.*



Remark 10. If $M' \not\subseteq M$ we have to require in addition that the pushout-pullback decomposition (Thm. 2 It. 2) holds also with $l \in M$ and $w \in M'$ in diagram (1)+(2) of Thm. 2.

Proof. With the E'-M' pair factorization for m_1 and m_2 we get an object K and morphisms $m : K \to G \in M'$, $o_1 : L_1 \to K$ and $o_2 : L_2 \to K$ with $(o_1, o_2) \in E'$ such that $m_1 = m \circ o_1$ and $m_2 = m \circ o_2$. We can now build the following extension diagram. First we construct the pullback over q_1 and m, derive the morphism t_1 and by applying Thm. 2 It. 2 both squares are pushouts. In the case $M' \not\subseteq M$ of Rem. 10 we have the pushout-pullback decomposition because $l_1 \in M$ and $m \in M'$. With Def. 3 we can build the pushout over r_1 and t_1, derive the morphism z_1 and with pushout decomposition this square is a pushout. The same construction is applied for the second transformation.

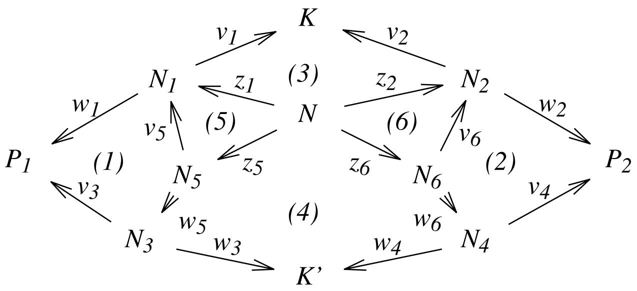


$P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ are non-parallel independent. Otherwise there are morphisms $i : L_1 \to N_2$ and $j : L_2 \to N_1$ with $v_2 \circ i = o_1$ and $v_1 \circ j = o_2$. Then $q_2 \circ s_2 \circ i = m \circ v_2 \circ i = m \circ o_1 = m_1$ and $q_1 \circ s_1 \circ j = m \circ v_1 \circ j = m \circ o_2 = m_2$, that means $H_1 \overset{p_1,m_1}{\Longleftarrow} G \overset{p_2,m_2}{\Longrightarrow} H_2$ □
are parallel independent, contradiction.

From [12] in the case of hypergraph transformation it is known already that confluence of critical pairs is not sufficient to show local confluence in general. In fact, we need a slightly stronger property, called strict confluence.

Definition 11 (strict confluence of critical pairs). *A critical pair $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ is called strictly confluent, if we have*

1. *confluence: the critical pair is confluent, i.e. there are transformations $P_1 \overset{*}{\Rightarrow} K'$, $P_2 \overset{*}{\Rightarrow} K'$ with derived spans $der(P_i \overset{*}{\Rightarrow} K') = P_i \overset{v_{i+2}}{\leftarrow} N_{i+2} \overset{w_{i+2}}{\rightarrow} K'$ for $i = 1, 2$.*
2. *strictness: Let $der(K \overset{p_i,o_i}{\Longrightarrow} P_i) = K \overset{v_i}{\leftarrow} N_i \overset{w_i}{\rightarrow} P_i$ $(i = 1, 2)$ and N_5, N_6 and N pullback objects of pullbacks (1), (2) resp. (3) then there are morphisms z_5 and z_6 such that (4), (5) and (6) commute.*



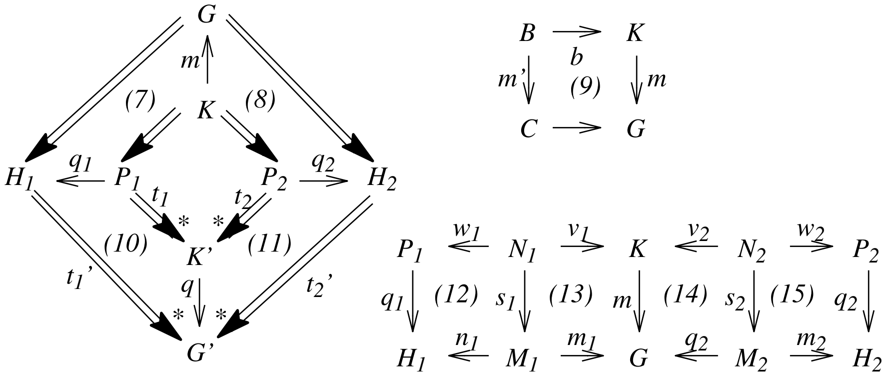
Remark 11. The strictness condition is a combination of corresponding conditions stated in [12] and [9]. More precisely, commutativity of (4) is required in [12] and that of (5) and (6) in [9]. In [12] however, commutativity of (5) and (6) seems to be a consequence of inclusion properties. The intuitive idea of strictness is that the common part N, which is preserved by each transformation of the critical pair, is also preserved by the transformations $P_1 \overset{*}{\Rightarrow} K'$ and $P_2 \overset{*}{\Rightarrow} K'$ and mapped by the same morphism $N \to K'$.

Finally our last main result states that strict confluence of all critical pairs implies local confluence. This result is also known as Critical Pair Lemma (see [12, 9]).

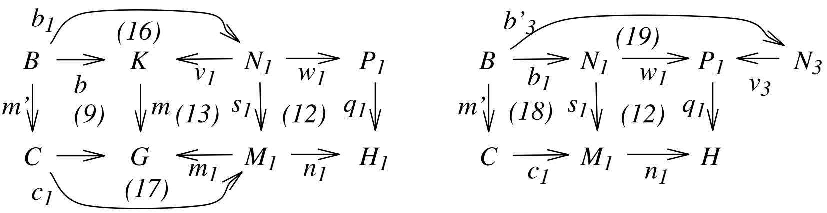
Theorem 6 (Local Confluence Theorem - Critical Pair Lemma). *An adhesive HLR system with E'-M' pair factorization, $M' \subseteq M$ and initial pushouts over M'-morphisms is locally confluent, if all its critical pairs are strictly confluent.*

Remark 12. See Rem. 10 for the case $M' \not\subseteq M$. In the proof we need that M is closed under decomposition (see Def. 3 It. 1) in order to show $b_3 \in M$.

Proof. Given a pair of direct transformations $H_1 \overset{p_1,m_2}{\Leftarrow} G \overset{p_2,m_2}{\Rightarrow} H_2$ we have to show the existence of transformations $t_1' : H_1 \overset{*}{\Rightarrow} G'$ and $t_2' : H_2 \overset{*}{\Rightarrow} G'$. If the given pair is parallel independent this follows from the local Church-Rosser theorem. If the given pair is not parallel independent Lem. 2 implies the existence of a critical pair $P_1 \overset{p_1,o_1}{\Longleftarrow} K \overset{p_2,o_2}{\Longrightarrow} P_2$ with extension diagrams (7) and (8) and $m \in M'$. By assumption this critical pair is strictly confluent leading to transformations $t_1 : P_1 \overset{*}{\Rightarrow} K'$, $t_2 : P_2 \overset{*}{\Rightarrow} K'$ and the diagram in Def. 11.



Now let (9) be an initial pushout over $m \in M'$ and consider the double pushouts (12), (13) and (14), (15) corresponding to extension diagrams (7) and (8) respectively.



Initiality of (9) applied to pushout (13) leads to unique $b_1,\ c_1 \in M$ such that (16) and (17) commute and (18) is pushout. By Lem. 1 (18) is initial pushout over s_1. Dually we obtain $b_2, c_2 \in M$ with $v_2 \circ b_2 = b$. Using pullback property of (3) in Def. 11 we obtain a unique $b_3 : B \to N$ with $z_1 \circ b_3 = b_1$ and $z_2 \circ b_3 = b_2$. Moreover $b_1,\ z_1 \in M$ implies $b_3 \in M$ by decomposition property of M. In order to show consistency of q_1 w.r.t. t_1 we have to construct $b_3' \in M$ such that (19) commutes where (18)+(12) is initial pushout over q_1 by Lem. 1. In fact $b_3' = w_5 \circ z_5 \circ b_3 \in M$ makes (19) commutative using (5) in Def. 11.

Dually q_2 is consistent w.r.t. t_2 using $b_4' = w_6 \circ z_6 \circ b_3 \in M$ and (6) in Def. 11. By the Embedding Theorem we obtain extension diagrams (10) and (11), where the morphism $q : K' \to G'$ is the same in both cases. This equality can be shown using part 3 of the Extension Theorem, where q is determined by an initial pushout of $m' : B \to C$ and $w_3 \circ b_3' : B \to K'$ in the first and $w_4 \circ b_4' : B \to K'$ in the second case and we have $w_3 \circ b_3' = w_4 \circ b_4'$ using commutativity of (4) in Def. 11. □

7 Conclusion

In this paper we have introduced adhesive HLR categories and systems combining the framework of adhesive categories in [3] and of HLR systems in [1]. We claim that this new framework is most important for different theories of graphs and graphical structures in computer science, which are mainly based on pushout constructions. As shown in this paper this includes first of all the double pushout approach in the theory of graph transformation and HLR systems [2, 1, 5], where important new results have been presented in this framework which are already applied to typed attributed graph transformation in [14]. Constraints and application conditions for DPO-transformations of adhesive HLR systems are considered already in [16]. On the other hand pushouts have also been used in semantics in order to derive well-behaved labeled transition systems by Leifer and Milner in [17], by Sassone and Sobociński in [18] and by König and Ehrig in [19]. We agree with [3] that the role of adhesive categories - and even more adhesive HLR categories - for this kind of applications is most likely to become comparable to the role of cartesian closed categories for simply typed lambda calculi as pointed out by Lambek and Scott in [20].

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