Graph Transformation in Adhesive HLR Categories

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Abstract. In this paper we introduce the categorical framework for rule-based transformations of high-level structures, e.g. graphs, hypergraphs, typed and attributed graphs, Petri nets, etc. based on adhesive high-level replacement (HLR) categories. This generalizes the classical theory of algebraic graph transformation systems. In particular we analyze the gluing condition for transformations in a categorical way and illustrate it with an example of Petri nets.

1 Introduction

We extend the concept of graph transformation from (Ehrig & Prange 2005) to a categorical framework in the sense of (Ehrig et al. 2004). In this section we review categories as well as categorical notions like monomorphisms and pushouts.

A *category* is a mathematical structure that has objects and morphisms, with an associative composition operation on the morphisms and an identity morphism for each object. Basic examples for categories are the category **Sets** of sets and functions and the category **Graphs** of graphs and graph morphisms.

In the categorical framework, the morphism class of monomorphisms is of special interest. In **Sets** and **Graphs** it corresponds to the class of all injective functions resp. graph morphisms.

Definition 1 (monomorphism) Given a category **C**, a morphism $m : B \to C$ is called a *monomorphism*, if for all morphisms $f, g: A \to B \in Mor_C$ holds: $m \circ f = m \circ g \Rightarrow f = g$.

For the application of transformation rules to an object we need a technique to glue objects together along a common subobject. The idea of a pushout generalizes this gluing construction in the sense of category theory.

Definition 2 (pushout) Given morphisms $f : A \to B$ and $g : A \to C$ in a category **C**. A *pushout* (D, f', g') over f and g (see diagram (1) below) is defined by a pushout object D and morphisms $f' : C \to D$ and $g' : B \to D$ with $f' \circ g = g' \circ f$, such that the following universal property is fulfilled:
For all objects X and morphisms $h : B \to X$ and $k : C \to X$ with $k \circ g = h \circ f$ there is a unique morphism $x : D \to X$ such that $x \circ g' = h$ and $x \circ f' = k$.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "g'"] \arrow[ddr, "h"] & \\
C \arrow[r, "f'"] \arrow[drr, "k"'] & D \arrow[dr, "x"] & \\
 & & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[dr, "x"] \arrow[drr, "h"] \arrow[ddr, "k"'] & & \\
 & A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "(2)"] & B \arrow[d, "g'"] \\
 & C \arrow[r, "f'"] & D
\end{tikzcd}
$$

Later we need the construction of a pullback, that is dual to a pushout and can be obtained by reversing all arrows (see diagram (2) above).

2 Adhesive and Weak Adhesive HLR Categories and Systems

In this section we introduce adhesive HLR categories based on the notion of van Kampen (VK) squares, which are the basis for adhesive HLR systems as rule-based transformation systems in the double pushout approach. Adhesive HLR categories are an extension of the concept of adhesive categories introduced in (Lack & Sobociński 2004).

The idea of a VK square is that of a pushout which is stable under pullbacks, and that pushout preservation vice versa implies pullback stability.

Definition 3 (van Kampen square) A pushout (1) is a *van Kampen square*, if for any commutative cube (2) with (1) in the bottom and the back faces being pullbacks holds: the top face is a pushout ⇔ the front faces are pullbacks.

$$
\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "g"] \\
C \arrow[r, "n"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & A' \arrow[dll, "f'"'] \arrow[dr, "m'"] \arrow[dd, "a"] & \\
C' \arrow[dr, "n'"] \arrow[dd, "c"'] & & & B' \arrow[dll, "g'"'] \arrow[dd, "b"] \\
 & D' \arrow[dd, "d"] & A \arrow[dll, "f"'] \arrow[dr, "m"] & \\
C \arrow[dr, "n"'] & & & B \arrow[dll, "g"] \\
 & D & &
\end{tikzcd}
\quad (2)
$$

For an adhesive HLR category we consider a distinguished class $\mathcal{M}$ of monomorphisms, so that pushouts along $\mathcal{M}$-morphisms have to be VK squares.

Definition 4 (adhesive HLR category) A category **C** with a morphism class $\mathcal{M}$ is called an *adhesive HLR category*, if

- $\mathcal{M}$ is a class of monomorphisms closed under isomorphisms, composition ($f : A \to B \in \mathcal{M}$, $g: B \to C \in \mathcal{M} \Rightarrow g \circ f \in \mathcal{M}$) and decomposition ($g \circ f \in \mathcal{M}$, $g \in \mathcal{M} \Rightarrow f \in \mathcal{M}$),
- **C** has pushouts and pullbacks along $\mathcal{M}$-morphisms (i.e. at least one of the given morphisms is in $\mathcal{M}$) and $\mathcal{M}$-morphisms are closed under pushouts and pullbacks,
- pushouts in **C** along $\mathcal{M}$-morphisms are VK squares.

For a slightly weaker version, called *weak adhesive HLR category*, see (Ehrig et al. 2005).

Typical examples of adhesive HLR categories are **Sets** and **Graphs** with the class $\mathcal{M}$ of all monomorphisms. Another interesting example are Petri nets (see Ex. 6).

In general, an adhesive HLR system is based on an adhesive HLR category. It has productions, also called rules that describe in an abstract way how objects in this system can be transformed. An application of a production is called *direct transformation* and describes how an object is actually changed by the production.

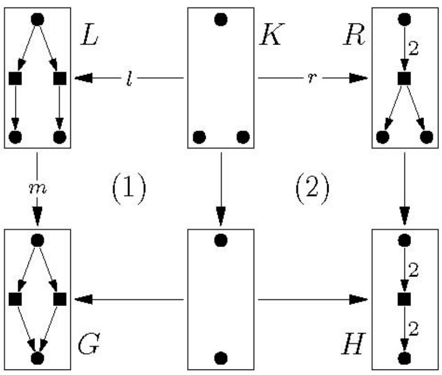
Definition 5 (adhesive HLR system, grammar and language) Given an adhesive HLR category $(\mathbf{C}, \mathcal{M})$, a *production* $p = (L \xleftarrow{l} K \xrightarrow{r} R)$ consists of three objects L, K and R called left hand side, gluing object and right hand side respectively, and morphisms $l : K \rightarrow L$, $r : K \rightarrow R$ with $l, r \in \mathcal{M}$. For p and an object G with a morphism $m : L \rightarrow G$, called match, a *direct transformation* $G \overset{p,m}{\Longrightarrow} H$ from G to an object H is given by the following diagram, where (1) and (2) are pushouts. A sequence $G_0 \Rightarrow G_1 \Rightarrow \ldots \Rightarrow G_n$ of direct transformations is called a *transformation* and is denoted as $G_0 \overset{*}{\Rightarrow} G_n$.

$$\begin{tikzcd}
L \arrow[d, "m"'] \arrow[dr, phantom, "(1)"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] \arrow[dr, phantom, "(2)"] & R \arrow[d, "n"] \\
G & D \arrow[l, "f"] \arrow[r, "g"'] & H
\end{tikzcd}$$

An *adhesive HLR system AHS* = $(\mathbf{C}, \mathcal{M}, P)$ consists of an adhesive HLR category $(\mathbf{C}, \mathcal{M})$ and a set of productions P. An *adhesive HLR grammar AHG* = (AHS, S) is an adhesive HLR system together with a distinguished start object S. The *language L* of *AHG* is defined by
$L = \{G \mid \exists \text{ transformation } S \overset{*}{\Rightarrow} G\}$.

In the case $(\mathbf{C}, \mathcal{M})$ = (**Graphs**, Monos) adhesive HLR systems are graph grammars in the sense of (Ehrig 1979).

Example 6 (Petri nets) A Petri net $N = (P, T, pre, post)$, also called place/transition net, consists of a set of places P, a set of transitions T and pre and post functions $pre, post: T \rightarrow P^{\oplus}$, where $P^{\oplus}$ is the free commutative monoid over P. A Petri net morphism $f = (f_P, f_T): N_1 \rightarrow N_2$ with $N_i = (P_i, T_i, pre_i, post_i)$ for $i = 1, 2$ consists of functions $f_P : P_1 \rightarrow P_2$ and $f_T : T_1 \rightarrow T_2$ such that $pre_2 \circ f_T = f_P^{\oplus} \circ pre_1$ and $post_2 \circ f_T = f_P^{\oplus} \circ post_1$.



The diagram on the right shows in the top a production p, where two transitions with the same place as their predomains are replaced by a single one. Round nodes symbolize places, square node transitions and the arrows visualize the pre and post functions. The morphisms are not explicitly shown, but are implied by the positions of the nodes.

The whole diagram shows the direct transformation for the given match m, where the Petri net G is transformed to H.

Petri nets and Petri net morphisms form the category **PTNets**, and (**PTNets**, $\mathcal{M}$) with the class $\mathcal{M}$ of all monomorphisms (i.e. morphisms, where both components are injective) is a weak adhesive HLR category. □

3 Initial Pushouts and Gluing Condition

In this section we introduce initial pushouts and the gluing condition for a production p via a match m. The main result states that in an adhesive HLR category with initial pushouts, p is applicable via m if and only if the gluing condition is fulfilled.

An initial pushout formalizes the construction of the boundary and context in (Ehrig 1979). For a morphism $f : A \to A'$ we want to construct a boundary $b : B \to A$, a boundary object B and a context C leading to a pushout. Roughly spoken, A' is the gluing of f and C along the boundary.

Definition 7 (initial pushout) Given $f : A \to A'$, a morphism $b : B \to A$ with $b \in \mathcal{M}$ is called the *boundary* over f if there is a pushout complement of f and b such that (1) is an *initial pushout* over f. Initiality of (1) over f means, that for every pushout (2) with $b' \in \mathcal{M}$ there exist unique morphisms $b^* : B \to D$ and $c^* : C \to E$ with $b^*, c^* \in \mathcal{M}$ such that $b' \circ b^* = b$, $c' \circ c^* = c$ and (3) is a pushout. Then B is called the *boundary object* and C the context with respect to f.

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d] \arrow[dr, phantom, "(1)"] & A \arrow[d, "f"] \\
C \arrow[r, "c"] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "b^*"] \arrow[d] \arrow[rr, bend left, "b"] \arrow[dr, phantom, "(3)"] & D \arrow[r, "b'"] \arrow[d] \arrow[dr, phantom, "(2)"] & A \arrow[d, "f"] \\
C \arrow[r, "c^*"] \arrow[rr, bend right, "c"'] & E \arrow[r, "c'"] & A'
\end{tikzcd}
$$

Intuitively, the gluing condition states that the boundary has to be preserved by the production.

Definition 8 (gluing condition) Given an adhesive HLR system *AHS* over an adhesive HLR category with initial pushouts. Then a match $m : L \to G$ satisfies the gluing condition w.r.t. a production $p = (L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R)$, if for the initial pushout (1) over m there is a morphism $b^* : B \to K \in \mathcal{M}$ such that $l \circ b^* = b$.

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d] \arrow[rr, bend left, "b^*"] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] & K \arrow[l, "l"'] \arrow[r, "r"] & R \\
C \arrow[r, "c"] & G & &
\end{tikzcd}
$$

Theorem 9 (existence and uniqueness of contexts) Given an adhesive HLR system *AHS* over an adhesive HLR category with initial pushouts. A match m: $L \to G$ satisfies the gluing condition w.r.t a production $p = (L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R)$ if and only if the context object D exists, i.e. there is a pushout complement (2) over $m \circ l$. If it exists, the context object D is unique up to isomorphism.

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d] \arrow[rr, bend left, "b^*"] \arrow[dr, phantom, "(1)"] & L \arrow[d, "m"] \arrow[dr, phantom, "(2)"] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \\
C \arrow[r, "c"] \arrow[rr, bend right, "c^*"'] & G & D \arrow[l, "f"'] &
\end{tikzcd}
$$

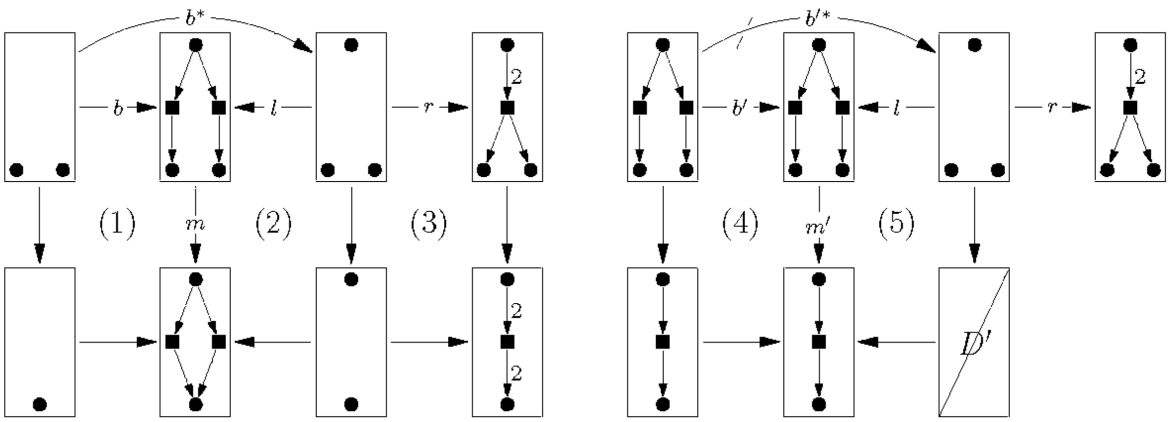
As a result of this theorem, if m w.r.t. p satisfies the gluing condition, then the context object D exists and we can apply p to G via m leading to a direct transformation $G \leftarrow D \to H$ (the second pushout exists due to the existence of the pushout along the $\mathcal{M}$-morphism r in an adhesive HLR category).

PROOF If the gluing condition is fulfilled, then we construct the pushout (3) with the pushout object D and morphisms k and c^* over $b^* \in \mathcal{M}$ and $B \to C$. This new pushout (3) together with morphisms c and $m \circ l$ implies a unique morphism f with $f \circ c^* = c$ and $m \circ l = f \circ k$, and by pushout decomposition of (3) also (2) is a pushout leading to the context object D.

If the context object D with the pushout (2) exists, the initiality of pushout (1) implies the existence of b^* with $l \circ b^* = b$.

The uniqueness of D follows from the uniqueness of pushout complements in adhesive HLR categories (see (Ehrig et al. 2004)). ■

Example 10 (gluing condition and context) On the left hand side of the following diagram we show the transformation from Ex. 6. The initial pushout (1) over m is shown and also the morphism b^* with $l \circ b^* = b$. That means the gluing condition is satisfied, therefore the context object D and the whole transformation exist. On the right hand side we see another match m' and the initial pushout (4) over m'. But we have no morphism b'^* and also no context object D' such that (5) becomes a pushout, i.e. p is not applicable to G via m.



□

References

Ehrig, H. (1979), Introduction to the Algebraic Theory of Graph Grammars (A Survey), *in* V. Claus, H. Ehrig & G. Rozenberg, eds, 'Graph Grammars and their Application to Computer Science and Biology', Vol. 73 of *LNCS*, Springer, pp. 1–69.

Ehrig, H., Ehrig, K., Prange, U. & Taentzer, G. (2005), *Fundamentals of Algebraic Graph Transformation*, Springer. (To appear).

Ehrig, H., Habel, A., Padberg, J. & Prange, U. (2004), Adhesive High-Level Replacement Categories and Systems, *in* H. Ehrig, G. Engels, F. Parisi-Presicce & G. Rozenberg, eds, 'Proceedings of ICGT 2004', Vol. 3256 of *LNCS*, Springer, pp.144–160.

Ehrig, H. & Prange, U. (2005), Modeling with Graph Transformations, *in* 'Proceedings of InterSymp 2005'. (To appear).

Lack, S. & Sobociński, P. (2004), Adhesive Categories, *in* I. Walukiewicz, ed., 'Proceedings of FOSSACS 2004', Vol. 2987 of *LNCS*, Springer, pp. 273–288.