Adhesive High-Level Replacement Systems with Negative Application Conditions

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Abstract

The goal of this paper is the generalization of basic results for adhesive High-Level Replacement (HLR) systems to adhesive HLR systems with negative application conditions. These conditions restrict the application of a rule by expressing that a specific structure should not be present before or after applying the rule to a certain context. Such a condition influences thus each rule application or transformation and therefore changes significantly the properties of the replacement system. The effect of negative application conditions on the replacement system is described in the generalization of the following results, formulated already for adhesive HLR systems without negative application conditions: Local Church-Rosser Theorem, Parallelism Theorem, Completeness Theorem for Critical Pairs, Concurrency Theorem, Embedding and Extension Theorem and Local Confluence Theorem or Critical Pair Lemma. These important generalized results will support the development of formal analysis techniques for adhesive HLR replacement systems with negative application conditions.

Keywords: Negative Application Conditions, Adhesive High-Level Replacement Categories

1 Introduction

Adhesive High-Level Replacement (HLR) categories as introduced in [3] provide a formal method to describe transformation systems. The resulting framework is called adhesive HLR systems. These systems are based on rules that describe in an abstract way how objects in adhesive HLR categories can be transformed. In [3], it is explained moreover how to define application conditions for rules that restrict the application of a rule. Most of the theoretical results in [3] though have been formulated for adhesive HLR systems based on rules without application conditions. These results should thus be generalized to adhesive HLR systems based on rules holding application conditions. The most frequently used kind of application condition is the so-called negative application condition (NAC) as introduced in [4] and used e.g. in [1,5,7,12,15]. It forbids a certain structure to be present before or after

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applying a rule. Therefore at first we concentrate on generalizing the theoretical results formulated for adhesive HLR systems based on rules without application conditions to adhesive HLR systems based on rules holding NACs. Shortly, we will speak about adhesive HLR systems with NACs.

Some important theoretical results for the particular case of graph transformation with NACs have been presented already in [10]. The overall goal is to come up with practical techniques for conflict detection and analysis [11,9] in transformation systems. In practice though most of these results are needed for the instantiation of typed attributed graph transformation systems with application conditions. This more general kind of graph transformation technique is most significant for modeling and metamodeling in software engineering and visual languages. Therefore the availability of all results for typed attributed graph transformation with NACs is an important motivation for the generalization to adhesive HLR systems with NACs. In [3], it has been already proven that such a typed attributed graph transformation system is a valid instantiation of adhesive HLR systems. Moreover results within adhesive HLR systems can be applied to all other instantiations of adhesive HLR systems such as e.g. hypergraph, algebraic signature or specification transformations with NACs.

At first we will generalize the local Church-Rosser property to transformations with NACs. Therefore a new notion of parallel independence and thus also sequential independence is defined between transformations with NACs. This is because it can not only happen that a transformation deletes a structure that is used by the second transformation as considered in the case without NACs. Moreover we should consider the case of the first transformation producing a structure which is forbidden by the second one. This situation is formalized in a new characterization of conflicts between transformations holding NACs. Moreover it is possible to formulate a Parallelism Theorem for transformations with NACs expressing how to summarize a sequence of two sequentially independent transformations into one parallel transformation step with the same effect.

In order to come up with techniques for conflict detection and analysis it is central to describe the theory for the so-called critical pairs. The notion of critical pairs is introduced in the area of term rewriting systems [6] and, later, introduced in the area of graph transformation for hypergraph rewriting [13,14] and then for all kinds of transformation systems fitting into the framework of adhesive HLR categories. Critical pairs describe conflicts in a minimal context. By means of the conflict characterization for transformations with NACs it is possible to define a generalized critical pair notion for adhesive HLR systems with NACs. Critical pairs should fulfill two important properties that were fulfilled in the case of adhesive HLR systems without NACs as well. It should be possible to formulate a Completeness Theorem and a so-called Critical Pair Lemma or Local Confluence Theorem. The Completeness Theorem describes that for every pair of transformations in conflict there exists a critical pair expressing the same conflict in a minimal context. We prove that the critical pair definition for adhesive HLR systems with NACs is complete in this sense.

In order to formulate and prove the Local Confluence Theorem for adhesive HLR systems with NACs we first need to generalize some other important results.

The first one is the Concurrency Theorem and the other ones are the Embedding and Extension Theorems. For the Concurrency Theorem it is explained how to construct a concurrent rule holding NACs from a sequence of rules holding NACs. The Concurrency Theorem with NACs states that the concurrent rule with NACs is applicable with the same result if and only if the rule sequence with NACs is applicable. The construction of the concurrent rule itself is analog to the case without NACs. It is necessary though to translate all NACs occurring in the rule sequence into equivalent NACs on the concurrent rule. Therefore we will use results for application conditions already described in [3] and some new results. Using the Concurrency Theorem it is not too difficult to find an extended condition on the extension morphism in order to generalize also the Embedding and Extension Theorem to adhesive HLR systems with NACs. Finally we are also able to handle critical pairs and local confluence with NACs. However the Local Confluence Theorem for adhesive HLR systems with NACs is not yet fully satisfactory.

2 Adhesive HLR systems with NACs

First we repeat the definition for an adhesive HLR category as introduced in [3].

Definition 2.1 [adhesive HLR category] A category **C** with a morphism class $\mathcal{M}$ is called an *adhesive HLR category*, if

(i) $\mathcal{M}$ is a class of monomorphisms closed under isomorphisms, composition and decomposition,

(ii) **C** has pushouts (PO) and pullbacks (PB) along $\mathcal{M}$-morphisms and $\mathcal{M}$-morphisms are closed under pushouts and pullbacks,

(iii) pushouts in **C** along $\mathcal{M}$-morphisms are Van Kampen squares.

Remark 2.2 Note that all results formulated in this paper will be applicable as well in weak adhesive HLR categories with NACs such as e.g. for Petri Net transformations with NACs.

For an *adhesive HLR category with NACs* we need in addition to an adhesive HLR category without NACs some additional properties on the special morphism classes in the category in order to be able to generalize all results. We distinguish three classes of morphisms, namely $\mathcal{M}, \mathcal{M}'$ and $\mathcal{Q}$, and a class of pairs of morphisms $\mathcal{E}'$. $\mathcal{M}$ is a subset of the class of all monomorphisms as given in [3] and the rule morphisms are always in $\mathcal{M}$. The non-existing morphism q in Def. 2.6 for negative application conditions is an element of the morphism set $\mathcal{Q}$. For pair factorization in Def. 5.25 in [3] we need moreover the classes $\mathcal{M}'$ and $\mathcal{E}'$. For the case of graph transformation systems with NACs with injective matching we can take $\mathcal{Q} = \mathcal{M}' = \mathcal{M}$, where $\mathcal{M}$ is the set of all graph monomorphisms and $\mathcal{E}'$ the set of jointly surjective pairs of graph morphisms. $\mathcal{M}, \mathcal{E}', \mathcal{M}'$ and $\mathcal{Q}$ should have the properties described in the following definition. Note that to each condition a remark is made in which theorem, lemma or definition this condition is needed for the first time.

Definition 2.3 [adhesive HLR category with NACs] An *adhesive HLR category with NACs* is an adhesive HLR category **C** with special morphism class $\mathcal{M}$ and in addition three morphism classes $\mathcal{M}', \mathcal{E}'$ and $\mathcal{Q}$ with the following properties:

- unique $\mathcal{E}' - \mathcal{M}'$ pair factorization (see Def. 5.25 in [3]) *needed for Theorem 4.5, Definition 5.8, Theorem 6.2*,
- epi - $\mathcal{M}$ factorization *needed for Lemma 5.2*,
- $\mathcal{M} - \mathcal{M}'$ PO-PB decomposition property (see Def. 5.27 in [3]) *needed for Theorem 4.4, Definition 5.8, Theorem 6.2*,
- $\mathcal{M} - \mathcal{Q}$ PO-PB decomposition property (see Def. 5.25 in [3]) *needed for Lemma 2.11*,
- initial PO over $\mathcal{M}'$ - morphisms (see Def. 6.1 in [3]) *needed for Theorem 6.3*,
- $\mathcal{M}'$ is closed under PO's and PB's along $\mathcal{M}$ - morphims *needed for Theorem 4.5, Definition 5.8, Theorem 6.3*,
- $\mathcal{Q}$ is closed under PO's and PB's along $\mathcal{M}$ - morphisms *needed for Lemma 2.11, Lemma 5.2*,
- induced PB-PO property for $\mathcal{M}$ and $\mathcal{Q}$ (see Def. 2.4) *needed for Lemma 5.2*,
- If $f : A \to B \in \mathcal{Q}$ and $g : B \to C \in \mathcal{M}'$ then $g \circ f \in \mathcal{Q}$.
  Composition property for morphisms in $\mathcal{M}'$ and $\mathcal{Q}$,
  needed for Theorem 4.4,
- If $g \circ f \in \mathcal{Q}$ and $g \in \mathcal{M}'$ then $f \in \mathcal{Q}$.
  Decomposition property for morphisms in $\mathcal{M}'$ and $\mathcal{Q}$,
  needed for Theorem 4.5,
- $\mathcal{Q}$ is closed under composition and decomposition *needed for Lemma 5.2, Lemma 2.11, Theorem 6.6*.

Note that these properties hold in particular for the case of graph transformation systems with NACs with $\mathcal{Q} = \mathcal{M}' = \mathcal{M}$, where $\mathcal{M}$ is the set of all graph monomorphisms and $\mathcal{E}'$ the set of jointly surjective pairs of graph morphisms.

Definition 2.4 [induced PB-PO property for $\mathcal{M}$ and $\mathcal{Q}$] Given $a : A \to C \in \mathcal{Q}$ and $b : B \to C \in \mathcal{M}$ and the following PB and PO

$$\begin{array}{ccc} D & \xrightarrow{d_2} & B \\ {\scriptstyle d_1}\downarrow & (PB) & \downarrow{\scriptstyle b} \\ A & \xrightarrow{a} & C \end{array} \qquad\qquad \begin{array}{ccc} D & \xrightarrow{d_2} & B \\ {\scriptstyle d_1}\downarrow & (PO) & \downarrow{\scriptstyle e_1} \\ A & \xrightarrow{e_2} & E \end{array}$$

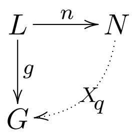
then the induced morphism $x : E \to C$ with $x \circ e_1 = b$ and $x \circ e_2 = a$ is a monomorphism in $\mathcal{Q}$.

Remark 2.5 Theorem 5.1 in [8] proves this property in adhesive categories for a,b being mono with the result that x is also mono.

A negative application condition or NAC as introduced in [4] forbids a certain structure to be present before or after applying the rule.

Definition 2.6 [negative application condition, rule with NACs]

- A *negative application condition* or $NAC(n)$ *on* L is an arbitrary morphism $n : L \to N$. A morphism $g : L \to G$ *satisfies* $NAC(n)$ on L i.e. $g \models NAC(n)$ if and only if $\nexists\, q : N \to G \in \mathcal{Q}$ such that $q \circ n = g$.

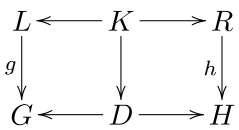


  A set of NACs on L is denoted by $NAC_L = \{NAC(n_i) | i \in I\}$. A morphism $g : L \to G$ *satisfies* NAC_L if and only if g satisfies all single NACs on L i.e. $g \models NAC(n_i)\ \forall i \in I$.

- A set of NACs NAC_L (resp. NAC_R) on L (resp. R) for a rule $p : L \overset{l}{\leftarrow} K \overset{r}{\rightarrow} R$ (with $l, r \in \mathcal{M}$) is called *left* (resp. *right*) NAC on p. $NAC_p = (NAC_L, NAC_R)$, consisting of a set of left and a set of right NACs on p is called a *set of NACs on* p. A *rule* (p, NAC_p) *with NACs* is a rule with a set of NACs on p.

Definition 2.7 [adhesive HLR system with NACs]

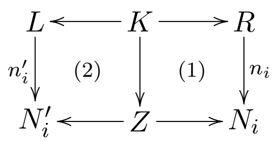
- An *adhesive HLR system with NACs* $AHS = (\mathbf{C}, \mathcal{M}, \mathcal{M}', \mathcal{E}', \mathcal{Q}, P)$ consists of an adhesive HLR category with NACs $(\mathbf{C}, \mathcal{M}, \mathcal{M}', \mathcal{E}', \mathcal{Q})$ and a set of rules with NACs P.
- A *direct transformation* $G \overset{p,g}{\Rightarrow} H$ via a rule $p : L \leftarrow K \rightarrow R$ with $NAC_p = (NAC_L, NAC_R)$ and a match $g : L \to G$ consists of the double pushout [2] (DPO)



  where g satisfies NAC_L, written $g \models NAC_L$ and $h : R \to H$ satisfies NAC_R, written $h \models NAC_R$. Since pushouts along $\mathcal{M}$-morphisms in an adhesive HLR category always exist, the DPO can be constructed if the pushout complement of $K \to L \to G$ exists. If so, we say that the match g satisfies the gluing condition of rule p. A *transformation*, denoted as $G_0 \overset{*}{\Rightarrow} G_n$, is a sequence $G_0 \Rightarrow G_1 \Rightarrow \cdots \Rightarrow G_n$ of direct transformations.

Remark 2.8 From now on we consider only adhesive HLR systems with rules having an empty set of right negative application conditions. This is without loss of generality, because each right NAC can be translated into an equivalent left NAC as explained in [3], where Definition 7.16 and Theorem 7.17 can be specialized to NACs as shown in the following construction and lemma.

Definition 2.9 [construction of left from right NACs] For each $NAC(n_i)$ on R with $n_i : R \to N_i$ of a rule $p = (L \leftarrow K \rightarrow R)$, the equivalent left application condition $L_p(NAC(n_i))$ is defined in the following way:



- If the pair $(K \to R, R \to N_i)$ has a pushout complement, we construct $(K \to Z, Z \to N_i)$ as the pushout complement (1). Then we construct pushout (2) with the morphism $n_i' : L \to N_i'$. Now we define $L_p(NAC(n_i)) = NAC(n_i')$.
- If the pair $(K \to R, R \to N_i)$ does not have a pushout complement, we define $L_p(NAC(n_i)) = true$.

For each set of NACs on R, $NAC_R = \cup_{i \in I} NAC(n_i)$ we define the following set of left NACs:

$$L_p(NAC_R) = \cup_{i \in I'} L_p(NAC(n_i'))$$

with $i \in I'$ if and only if the pair $(K \to R, R \to N_i)$ has a pushout complement.

Remark 2.10 Note that Z is unique since pushout complements along $\mathcal{M}$-morphisms are unique up to isomorphism in adhesive HLR categories.

Lemma 2.11 (equivalence of left and right NACs) *For every rule p with NAC_R a set of right NACs on p, $L_p(NAC_R)$ as defined in Definition 2.9 is a set of left NACs on p such that for all direct transformations $G \overset{p,g}{\Rightarrow} H$ with comatch h,*

$$g \models L_p(NAC_R) \Leftrightarrow h \models NAC_R$$

Proof The proof corresponds to case 1 and 3 in the proof of Theorem 7.17 in [3].□

Definition 2.12 [inverse rule with NACs] For a rule $p : L \leftarrow K \rightarrow R$ with $NAC_p = (NAC_L, \emptyset)$, the inverse rule is defined by $p^{-1} = R \leftarrow K \rightarrow L$ with $NAC_{p^{-1}} = (L_{p^{-1}}(NAC_L), \emptyset)$.

Theorem 2.13 (Inverse Direct Transformation with NACs) *For each direct transformation with NACs $G \Rightarrow H$ via a rule $p : L \leftarrow K \rightarrow R$ with NAC_p a set of left NACs on p, there exists an inverse direct transformation with NACs $H \Rightarrow G$ via the inverse rule p^{-1} with $NAC_{p^{-1}}$.*

Proof This follows directly from Def. 2.12 and Lemma 2.11. □

3 Parallelism in Adhesive HLR Systems with NACs

In order to generalize the notion of parallelism to adhesive HLR systems with NACs at first it is necessary to define when two direct transformations with NACs are parallel independent. For a pair of transformations with NACs it is not only possible that one transformation deletes a structure which is needed by the other one, but also that one transformation produces a structure which is forbidden by the other one. For this new notion of parallel independence it is possible to formulate the local Church-Rosser property with NACs and also a Parallelism Theorem with NACs as described in this section.

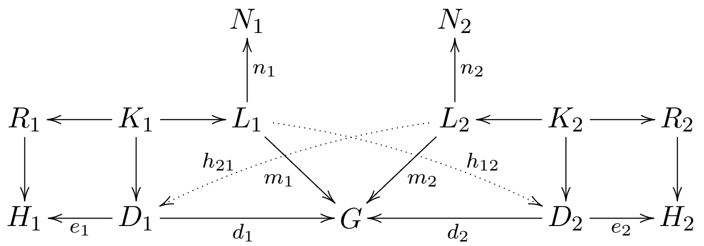
Definition 3.1 [parallel and sequential independence] Two direct transformations $G \overset{(p_1,m_1)}{\Longrightarrow} H_1$ with NAC_{p_1} and $G \overset{(p_2,m_2)}{\Longrightarrow} H_2$ with NAC_{p_2} are *parallel independent* if

$$\exists h_{12} : L_1 \to D_2 \ s.t. \ (d_2 \circ h_{12} = m_1 \ and \ e_2 \circ h_{12} \models NAC_{p_1})$$

and

$$\exists h_{21} : L_2 \to D_1 \ s.t. \ (d_1 \circ h_{21} = m_2 \ and \ e_1 \circ h_{21} \models NAC_{p_2})$$

as in the following diagram:



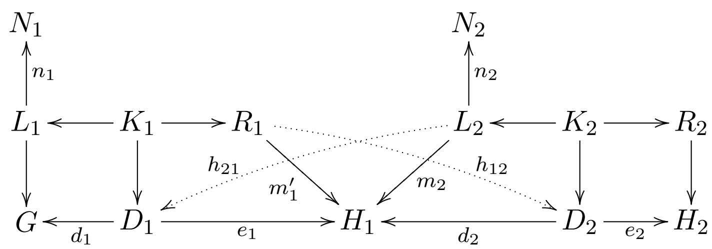
Two direct transformations $G \overset{(p_1,m_1)}{\Longrightarrow} H_1$ with NAC_{p_1} and $H_1 \overset{(p_2,m_2)}{\Longrightarrow} H_2$ with NAC_{p_2} are *sequentially independent* if

$$\exists h_{12} : R_1 \to D_2 \ s.t. \ (d_2 \circ h_{12} = m'_1 \ and \ e_2 \circ h_{12} \models NAC_{p_1^{-1}})$$

and

$$\exists h_{21} : L_2 \to D_1 \ s.t. \ (e_1 \circ h_{21} = m_2 \ and \ d_1 \circ h_{21} \models NAC_{p_2})$$

as in the following diagram:



Remark 3.2 Note that as for the case without NACs we have the following relationship between parallel and sequential independency: $G \overset{p_1}{\Rightarrow} H_1 \overset{p_2}{\Rightarrow} H_2$ are sequentially independent iff $G \overset{p_1^{-1}}{\Leftarrow} H_1 \overset{p_2}{\Rightarrow} H_2$ are parallel independent.

Theorem 3.3 (Local Church-Rosser Theorem with NACs) *Given an adhesive HLR system with NACs AHS and two parallel independent direct transformations with NACs* $H_1 \overset{p_1,m_1}{\Leftarrow} G \overset{p_2,m_2}{\Rightarrow} H_2$, *there are an object* G' *and direct transformations* $H_1 \overset{p_2,m'_2}{\Rightarrow} G'$ *and* $H_2 \overset{p_1,m'_1}{\Rightarrow} G'$ *such that* $G \overset{p_1,m_1}{\Rightarrow} H_1 \overset{p_2,m'_2}{\Rightarrow} G'$ *and* $G \overset{p_2,m_2}{\Rightarrow} H_2 \overset{p_1,m'_1}{\Rightarrow} G'$ *are sequentially independent. Vice versa, given two sequentially independent direct transformations with NACs* $G \overset{p_1,m_1}{\Rightarrow} H_1 \overset{p_2,m'_2}{\Rightarrow} G'$ *there are an object* H_2 *and sequentially independent direct transformations* $G \overset{p_2,m_2}{\Rightarrow} H_2 \overset{p_1,m'_1}{\Rightarrow} G'$

such that $H_1 \overset{p_1,m_1}{\Leftarrow} G \overset{p_2,m_2}{\Rightarrow} H_2$ *are parallel independent:*

$$\begin{array}{ccccc} & & H_1 & & \\ & \overset{p_1,m_1}{\nearrow} & & \overset{p_2,m_2'}{\searrow} & \\ G & & & & G' \\ & \underset{p_2,m_2}{\searrow} & & \underset{p_1,m_1'}{\nearrow} & \\ & & H_2 & & \end{array}$$

Proof

(i) Given the parallel independent transformations $H_1 \overset{p_1,m_1}{\Leftarrow} G \overset{p_2,m_2}{\Rightarrow} H_2$:

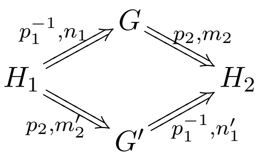
$$\begin{array}{ccccc} L_1 & \leftarrow & K_1 & \rightarrow & R_1 \\ {\scriptstyle m_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle n_1} \\ G & \underset{f_1}{\leftarrow} & D_1 & \underset{g_1}{\rightarrow} & H_1 \end{array} \qquad \begin{array}{ccccc} L_2 & \leftarrow & K_2 & \rightarrow & R_2 \\ {\scriptstyle m_2}\downarrow & & \downarrow & & \downarrow{\scriptstyle n_2} \\ G & \underset{f_2}{\leftarrow} & D_2 & \underset{g_2}{\rightarrow} & H_2 \end{array}$$

Because of Def. 3.1 and the parallel independence with NACs of $H_1 \overset{p_1,m_1}{\Leftarrow} G \overset{p_2,m_2}{\Rightarrow} H_2$ we know that there exists $i_2 : L_2 \to D_1$ (resp. $i_1 : L_1 \to D_2$) s.t. $f_1 \circ i_2 = m_2$ (resp. $f_2 \circ i_1 = m_1$) and moreover $g_1 \circ i_2 \models NAC_{p_2}$ (resp. $g_2 \circ i_1 \models NAC_{p_1}$). Because of the Local Church-Rosser Theorem for parallel independent transformations without NACs all necessary pushouts in $H_1 \overset{p_2}{\Rightarrow} G'$ and $H_2 \overset{p_1}{\Rightarrow} G'$ can be constructed s.t. $G \overset{p_1}{\Rightarrow} H_1 \overset{p_2}{\Rightarrow} G'$ and $G \overset{p_2}{\Rightarrow} H_2 \overset{p_1}{\Rightarrow} G'$ are sequentially independent according to Def. 5.9 in [3] for direct transformations without NACs. This means in particular that $t_1 : R_1 \to D_2'$ (resp. $t_2 : R_2 \to D_1'$) exist s.t. $s_1 \circ t_1 = n_1$ (resp. $s_2 \circ t_2 = n_2$) and the following pushout diagrams exist:

$$\begin{array}{ccccc} L_2 & \leftarrow & K_2 & \rightarrow & R_2 \\ {\scriptstyle g_1 \circ i_2}\downarrow & & \downarrow & & \downarrow{\scriptstyle g_1' \circ t_2} \\ H_1 & \overset{s_1}{\leftarrow} & D_2' & \overset{g_2'}{\rightarrow} & G' \end{array} \qquad \begin{array}{ccccc} L_1 & \leftarrow & K_1 & \rightarrow & R_1 \\ {\scriptstyle g_2 \circ i_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle g_2' \circ t_1} \\ H_2 & \overset{s_2}{\leftarrow} & D_1' & \overset{g_1'}{\rightarrow} & G' \end{array}$$

Since $g_1 \circ i_2 \models NAC_{p_2}$ and $g_2 \circ i_1 \models NAC_{p_1}$, $H_1 \overset{p_2}{\Rightarrow} G'$ and $H_2 \overset{p_1}{\Rightarrow} G'$ are valid direct transformations with NACs. For the sequential independence of $G \overset{p_1}{\Rightarrow} H_1 \overset{p_2}{\Rightarrow} G'$ we have to show that i_2, t_1 are the required morphisms. For i_2 we have $f_1 \circ i_2 = m_2$, and therefore $f_1 \circ i_2 \models NAC_{p_2}$ follows by assumption. Now we investigate $g_2' \circ t_1$. Because of Theorem 2.13 and the fact that $g_2 \circ i_1 \models NAC_{p_1}$ it follows directly that also $g_2' \circ t_1 \models NAC_{p_1^{-1}}$. Analogously the sequential independence of $G \overset{p_2}{\Rightarrow} H_2 \overset{p_1}{\Rightarrow} G'$ can be proven.

(ii) Given sequentially independent direct transformations with NACs $G \overset{p_1,m_1}{\Rightarrow} H_1 \overset{p_2,m_2'}{\Rightarrow} G'$ with comatches n_1' and n_2', respectively, from Remark 3.2 we obtain parallel independent direct transformations with NACs $G \overset{p_1^{-1},n_1}{\Leftarrow} H_1 \overset{p_2,m_2'}{\Rightarrow} G'$. Now part (i) of the proof gives us sequentially independent direct transformations with NACs $H_1 \overset{p_1^{-1},n_1}{\Rightarrow} G \overset{p_2,m_2}{\Rightarrow} H_2$ and $H_1 \overset{p_2,m_2'}{\Rightarrow} G' \overset{p_1^{-1},n_1'}{\Rightarrow} H_2$. Applying again Remark 3.2 to the first transformation we obtain parallel independent direct transformations with NACs $H_1 \overset{p_1,m_1}{\Leftarrow} G \overset{p_2,m_2}{\Rightarrow} H_2$:



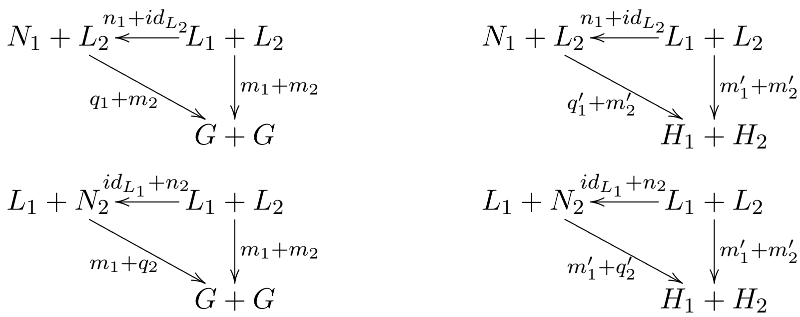
□

Definition 3.4 [parallel rule with NAC] Let $AHS = (\mathbf{C}, \mathcal{M}, \mathcal{M}', \mathcal{E}', \mathcal{Q}, P)$ be an adhesive HLR system with NACs, where $(\mathbf{C}, \mathcal{M}, \mathcal{Q})$ has binary coproducts compatible with $\mathcal{M}$. Given two rules $p_1 = (L_1 \overset{l_1}{\leftarrow} K_1 \overset{r_1}{\rightarrow} R_1)$ with NAC_{p_1} and $p_2 = (L_2 \overset{l_2}{\leftarrow} K_2 \overset{r_2}{\rightarrow} R_2)$ with NAC_{p_2}, the *parallel rule* $p_1 + p_2$ with $NAC_{p_1+p_2}$ is defined by the coproduct constructions over the corresponding objects and morphisms: $p_1 + p_2 = (L_1 + L_2 \overset{l_1+l_2}{\leftarrow} K_1 + K_2 \overset{r_1+r_2}{\rightarrow} R_1 + R_2)$ and $NAC_{p_1+p_2} = \{n_1 + id_{L_2} | n_1 \in NAC_{p_1}\} \cup \{id_{L_1} + n_2 | n_2 \in NAC_{p_2}\}$.

Remark 3.5 It is not sufficient as in the case without NACs to define a direct parallel transformation simply as the application of a parallel rule $p_1 + p_2$. For the case with NACs the application of a parallel rule with NACs is a direct parallel transformation with NACs if an extra condition, called NAC-compatibility, is satisfied. This condition expresses that the NACs on rule p_1 and p_2 are satisfied by all matches occuring in the direct transformations when sequentializing the direct parallel transformation.

Definition 3.6 [parallel transformation with NAC] Given a parallel rule $p_1 + p_2$ with $NAC_{p_1+p_2}$. Consider a direct transformation $G \Rightarrow G'$ via $p_1 + p_2$ with $NAC_{p_1+p_2}$ and a match $m : L_1 + L_2 \rightarrow G$. This transformation is *NAC-compatible* if and only if for each $n_1 \in NAC_{p_1}$ (resp. $n_2 \in NAC_{p_2}$) $\nexists\, q_1, q_1' \in \mathcal{Q}$ (resp. $\nexists\, q_2, q_2' \in \mathcal{Q}$) s.t.

- $(q_1 + m_2) \circ (n_1 + id_{L_2}) = m_1 + m_2$ (resp. $(m_1 + q_2) \circ (id_{L_1} + n_2) = m_1 + m_2$) and
- $(q_1' + m_2') \circ (n_1 + id_{L_2}) = m_1' + m_2'$ (resp. $(m_1' + q_2') \circ (id_{L_1} + n_2) = m_1' + m_2'$)



with m_1, m_2 the matches of the direct transformations $G \Rightarrow H_1$ and $G \Rightarrow H_2$ via p_1 resp. p_2 and m_1' and m_2' the matches of the direct transformations $H_2 \Rightarrow G'$ and $H_1 \Rightarrow G'$ via p_1 resp. p_2 as constructed in the Parallelism Theorem without NACs (Analysis part in Theorem 5.18 in [3]). A NAC-compatible direct parallel transformation via p_1+p_2 and m is called a *direct parallel transformation with NACs* or *parallel transformation with NACs*, for short.

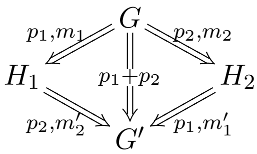
Theorem 3.7 (Parallelism Theorem with NACs) *Let $AHS = (\boldsymbol{C}, \mathcal{M}, \mathcal{M}', \mathcal{E}', \mathcal{Q}, P)$ be an adhesive HLR system with NACs, where $(\boldsymbol{C}, \mathcal{M})$*

has binary coproducts compatible with $\mathcal{M}$.

(i) Synthesis. *Given a sequentially independent direct transformation sequence with NACs $G \Rightarrow H_1 \Rightarrow G'$ via p_1, m_1 (resp. p_2, m_2') with NAC_{p_1} (resp. NAC_{p_2}), then there is a construction leading to a parallel transformation with NACs $G \Rightarrow G'$ via $[m_1, m_2]$ and the parallel rule $p_1 + p_2$ with $NAC_{p_1+p_2}$, called a* synthesis construction.

(ii) Analysis. *Given a direct parallel transformation with NACs $G \Rightarrow G'$ via $m : L_1 + L_2 \to G$ and the parallel rule $p_1 + p_2$ with $NAC_{p_1+p_2}$, then there is a construction leading to two sequentially independent transformation sequences with NACs $G \Rightarrow H_1 \Rightarrow G'$ via p_1, m_1 and p_2, m_2' and $G \Rightarrow H_2 \Rightarrow G'$ via p_2, m_2 and p_1, m_1', called an* analysis construction.

(iii) Bijective Correspondence. *The synthesis and analysis constructions are inverse to each other up to isomorphism.*



Proof

(i) Given the sequentially independent direct transformations with NACs $G \overset{p_1,m_1}{\Rightarrow} H_1 \overset{p_2,m_2}{\Rightarrow} G'$, using the Parallelism Theorem (Theorem 5.18 in [3]) without NACs we can construct the following double pushout:

$$\begin{array}{ccccc} L_1 + L_2 & \longleftarrow & K_1 + K_2 & \longrightarrow & R_1 + R_2 \\ {\scriptstyle [m_1,m_2]}\downarrow & & \downarrow & & \downarrow \\ G & \longleftarrow & D & \longrightarrow & G' \end{array}$$

Now we have to prove that this direct transformation is NAC-compatible. Note at first that because of Theorem 3.3 we have also the following two sequentially independent direct transformations with NACs $G \Rightarrow H_2 \Rightarrow G'$ via p_2, m_2 and p_1, m_1'. We now show that for each $n_1 \in NAC_{p_1}$ (resp. $n_2 \in NAC_{p_2}$) $\nexists\, q_1, q_1' \in \mathcal{Q}$ (resp. $\nexists\, q_2, q_2' \in \mathcal{Q}$) s.t.

- $(q_1 + m_2) \circ (n_1 + id_{L_2}) = m_1 + m_2$ (resp. $(m_1 + q_2) \circ (id_{L_1} + n_2) = m_1 + m_2$) and
- $(q_1' + m_2') \circ (n_1 + id_{L_2}) = m_1' + m_2'$ (resp. $(m_1' + q_2') \circ (id_{L_1} + n_2) = m_1' + m_2'$)

Suppose that q_1 would exist satisfying the first property. Then it follows in particular that $q_1 \circ n_1 = m_1$ and this means that $m_1 \not\models NAC_{p_1}$ which is a contradiction. Suppose moreover that q_1' exists satisfying the second property. Then it follows that $q_1' \circ n_1 = m_1'$ and this would mean that $m_1' \not\models NAC_{p_1}$ which is a contradiction as well. We can argue analogously for q_2 and q_2'.

(ii) Given a parallel transformation with NACs $G \overset{p_1+p_2,m}{\Longrightarrow} G'$ then because of the Parallelism Theorem (Theorem 5.18 in [3]) without NACs it follows that $G \Rightarrow H_1$ and $G \Rightarrow H_2$ are parallel independent without NACs and moreover the necessary double pushouts for $G \Rightarrow H_1 \Rightarrow G'$ via p_1, m_1 and p_2, m_2' and

$G \Rightarrow H_2 \Rightarrow G'$ via p_2, m_2 and p_1, m'_1 can be constructed s.t. they are sequentially independent without NACs. If we can prove that $m_1, m'_1 \models NAC_{p_1}$ and $m_2, m'_2 \models NAC_{p_2}$, then we know that $G \Rightarrow H_1$ and $G \Rightarrow H_2$ are parallel independent as transformations with NACs. We know that the direct transformation $G \stackrel{p_1+p_2,m}{\Rightarrow} G'$ is NAC-compatible. So we use the fact that for each $n_1 \in NAC_{p_1}$ (resp. $n_2 \in NAC_{p_2}$) $\nexists\, q_1, q'_1 \in \mathcal{Q}$ (resp. $\nexists\, q_2, q'_2 \in \mathcal{Q}$) s.t.

- $(q_1 + m_2) \circ (n_1 + id_{L_2}) = m_1 + m_2$ (resp. $(m_1 + q_2) \circ (id_{L_1} + n_2) = m_1 + m_2$) and
- $(q'_1 + m'_2) \circ (n_1 + id_{L_2}) = m'_1 + m'_2$ (resp. $(m'_1 + q'_2) \circ (id_{L_1} + n_2) = m'_1 + m'_2$)

Suppose that $m_1 \not\models NAC_{p_1}$. Then there exists some $q_1 \in Q$ s.t. $q_1 \circ n_1 = m_1$ with $n_1 \in NAC_{p_1}$. Then the first property for q_1 would hold and this is a contradiction. Suppose that $m'_2 \not\models NAC_{p_2}$. Then there exists some $q'_2 \in Q$ s.t. $q'_2 \circ n_2 = m'_2$ with $n_2 \in NAC_{p_2}$. Then the second property for q'_2 would hold and this is a contradiction. We can argue analogously for m_2 and m'_1.
Now we have proven that $G \Rightarrow H_1$ and $G \Rightarrow H_2$ are parallel independent with NACs and from Theorem 3.3 it follows that $G \Rightarrow H_1 \Rightarrow G'$ and $G \Rightarrow H_2 \Rightarrow G'$ are then sequentially independent with NACs.

(iii) Because of the uniqueness of pushouts and pushout complements, the constructions are inverse to each other up to isomorphism.

□

The following lemma allows an elegant characterization of conflicts (Def. 3.9) in Lemma 3.10.

Lemma 3.8 (Unique Match) *Given two direct transformations* $G \stackrel{(p_1,m_1)}{\Longrightarrow} H_1$ *with* NAC_{p_1} *and* $G \stackrel{(p_2,m_2)}{\Longrightarrow} H_2$ *with* NAC_{p_2}, *then the following holds:*

- *if* $\exists h_{12} : L_1 \to D_2$ *s.t.* $d_2 \circ h_{12} = m_1$ *then* h_{12} *is unique,*
- *if* $\exists h_{21} : L_2 \to D_1$ *s.t.* $d_1 \circ h_{21} = m_2$ *then* h_{21} *is unique.*

Proof Since each rule consists of two morphisms in $\mathcal{M}$ and $\mathcal{M}$-morphisms are closed under pushouts d_1 and d_2 are in $\mathcal{M}$ as well. Suppose there exists $h'_{12} : L_1 \to D_2 : d_2 \circ h'_{12} = m_1$ then because of d_2 in $\mathcal{M}$ and thus a monomorphism and $d_2 \circ h'_{12} = d_2 \circ h_{12} = m_1$ it follows that $h'_{12} = h_{12}$. Analogously one can prove that h_{21} is unique. □

Definition 3.9 [conflict] Two direct transformations $G \stackrel{(p_1,m_1)}{\Longrightarrow} H_1$ with NAC_{p_1} and $G \stackrel{(p_2,m_2)}{\Longrightarrow} H_2$ with NAC_{p_2} are *in conflict* if they are not parallel independent i.e. if

$$\nexists h_{12} : L_1 \to D_2 \ s.t.\ (d_2 \circ h_{12} = m_1 \ and\ e_2 \circ h_{12} \models NAC_{p_1})$$

or

$$\nexists h_{21} : L_2 \to D_1 \ s.t.\ (d_1 \circ h_{21} = m_2 \ and\ e_1 \circ h_{21} \models NAC_{p_2}).$$

In the following conflict characterization it is described which type of conflicts may arise between a pair of direct transformations $G \stackrel{(p_1,m_1)}{\Longrightarrow} H_1$ and $G \stackrel{(p_2,m_2)}{\Longrightarrow} H_2$. The so-called use-delete-conflict occurs when rule p_1 applied to H_2 wants to use some item which is deleted by p_2 applied to G leading to H_2. A forbid-produce-conflict occurs when rule p_1 applied to H_2 forbids some item according to NAC_{p_1} which is produced by p_2. Analogously we can have a delete-use- or produce-forbid-conflict.

Lemma 3.10 (Conflict Characterization) *Two direct transformations $G \overset{(p_1,m_1)}{\Longrightarrow} H_1$ with NAC_{p_1} and $G \overset{(p_2,m_2)}{\Longrightarrow} H_2$ with NAC_{p_2} are* in conflict *if and only if:*

(i) (a) $\nexists h_{12} : L_1 \to D_2 : d_2 \circ h_{12} = m_1$ *(use-delete-conflict)*
or
(b) *there exists a unique $h_{12} : L_1 \to D_2 : d_2 \circ h_{12} = m_1$, but $e_2 \circ h_{12} \not\models NAC_{p_1}$ (forbid-produce-conflict)*
or

(ii) (a) $\nexists h_{21} : L_2 \to D_1 : d_1 \circ h_{21} = m_2$ *(delete-use-conflict)*
or
(b) *there exists a unique $h_{21} : L_2 \to D_1 : d_1 \circ h_{21} = m_2$, but $e_1 \circ h_{21} \not\models NAC_{p_2}$ (produce-forbid-conflict).*

Proof $G \overset{(p_1,m_1)}{\Rightarrow} H_1$ with NAC_{p_1} and $G \overset{(p_2,m_2)}{\Rightarrow} H_2$ with NAC_{p_2} are *in conflict* if

$$\nexists h_{12} : L_1 \to D_2 \ s.t.\ (d_2 \circ h_{12} = m_1 \ and \ e_2 \circ h_{12} \models NAC_{p_1})$$

or

$$\nexists h_{21} : L_2 \to D_1 \ s.t.\ (d_1 \circ h_{21} = m_2 \ and \ e_1 \circ h_{21} \models NAC_{p_2})$$

We consider at first the first line of this disjunction. Let $A(h_{12}) := d_2 \circ h_{12} = m_1$, $B(h_{12}) := e_2 \circ h_{12} \models NAC_{p_1}$, $P(h_{12}) := (A(h_{12}) \wedge B(h_{12}))$ and $\mathcal{M}_{12}$ be the set of all morphisms from L_1 to D_2. Then the first line is equivalent to

$$\nexists h_{12} \in \mathcal{M}_{12} : (A(h_{12}) \wedge B(h_{12})) \equiv \nexists h_{12} \in \mathcal{M}_{12} : P(h_{12})$$

This is equivalent to

$$\forall h_{12} \in \mathcal{M}_{12} : \neg P(h_{12}) \equiv (\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg P(h_{12}))$$

Moreover $P \equiv A \wedge B \equiv A \wedge (A \Rightarrow B)$ and thus $\neg P \equiv \neg(A \wedge B) \equiv \neg(A \wedge (A \Rightarrow B)) \equiv \neg A \vee \neg(A \Rightarrow B) \equiv \neg A \vee \neg(\neg A \vee B) \equiv \neg A \vee (A \wedge \neg B)$. This implies that $(\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg P(h_{12})) \equiv$

$$(\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg A(h_{12}) \vee (A(h_{12}) \wedge \neg B(h_{12})))$$

Because of Lemma 3.8 and because the disjunction holding for each morphism in $\mathcal{M}_{12}$ is an exclusive one this is equivalent to

$$(\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg A(h_{12})) \vee (\exists! h_{12} \in \mathcal{M}_{12} : (A(h_{12}) \wedge \neg B(h_{12})))$$

Now $(\mathcal{M}_{12} = \emptyset) \vee (\mathcal{M}_{12} \neq \emptyset \wedge \forall h_{12} \in \mathcal{M}_{12} : \neg A(h_{12})) \equiv \forall h_{12} \in \mathcal{M}_{12} : \neg A(h_{12}) \equiv \nexists h_{12} \in \mathcal{M}_{12} : A(h_{12})$. This implies finally that $\nexists h_{12} : L_1 \to D_2 \ s.t.\ (d_2 \circ h_{12} = m_1 \ and \ e_2 \circ h_{12} \models NAC_{p_1})$ is equivalent to

$$(\nexists h_{12} \in \mathcal{M}_{12} : d_2 \circ h_{12} = m_1) \vee (\exists! h_{12} \in \mathcal{M}_{12} : (d_2 \circ h_{12} = m_1 \wedge e_2 \circ h_{12} \not\models NAC_{p_1}))$$

is equivalent to

(i) (a) $\nexists h_{12} : L_1 \to D_2 : d_2 \circ h_{12} = m_1$ (use-delete-conflict)
or
(b) there exists a unique $h_{12} : L_1 \to D_2 : d_2 \circ h_{12} = m_1$, but $e_2 \circ h_{12} \not\models NAC_{p_1}$ (forbid-produce-conflict)

Analogously we can proceed for the second part of the disjunction. □

Note that a use-delete-conflict (resp. delete-use-conflict) cannot occur simultaneously to a forbid-produce-conflict (resp. produce-forbid-conflict), since $(1.a) \Rightarrow \neg(1.b)$ (resp. $(2.a) \Rightarrow \neg(2.b)$). The following types of conflicts can occur simultaneously though: use-delete/delete-use-, use-delete/produce-forbid-, forbid-produce/delete-use-, forbid-produce/produce-forbid-conflicts.

4 Critical Pairs for adhesive HLR Systems with NACs

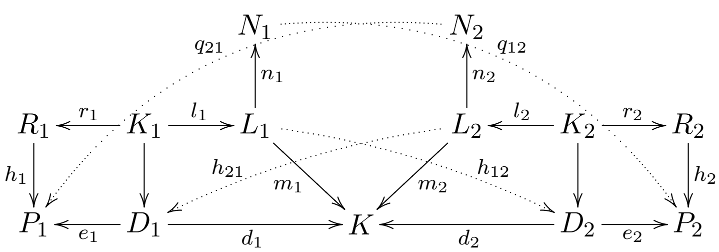
The conflict characterization formulated in the last section leads to a new critical pair notion for transformations with NACs. A critical pair describes a conflict between two rules in a minimal context. Therefore we come to the following critical pair definition with $\mathcal{E}'$ expressing this minimal context. Moreover, it is proven in this section that this critical pair definition satisfies completeness.

Definition 4.1 [critical pair] A critical pair is a pair of direct transformations $K \overset{(p_1,m_1)}{\Rightarrow} P_1$ with NAC_{p_1} and $K \overset{(p_2,m_2)}{\Rightarrow} P_2$ with NAC_{p_2} such that:

(i) (a) $\nexists h_{12} : L_1 \to D_2 : d_2 \circ h_{12} = m_1$ and (m_1, m_2) in $\mathcal{E}'$ (use-delete-conflict)
or
(b) there exists $h_{12} : L_1 \to D_2$ s.t. $d_2 \circ h_{12} = m_1$, but for one of the NACs $n_1 : L_1 \to N_1$ of p_1 there exists a morphism $q_{12} : N_1 \to P_2 \in \mathcal{Q}$ s.t. $q_{12} \circ n_1 = e_2 \circ h_{12}$ and (q_{12}, h_2) in $\mathcal{E}'$ (forbid-produce-conflict)

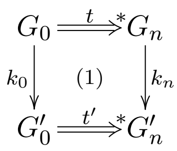
or

(ii) (a) $\nexists h_{21} : L_2 \to D_1 : d_1 \circ h_{21} = m_2$ and (m_1, m_2) in $\mathcal{E}'$ (delete-use-conflict)
or
(b) there exists $h_{21} : L_2 \to D_1$ s.t. $d_1 \circ h_{21} = m_2$, but for one of the NACs $n_2 : L_2 \to N_2$ of p_2 there exists a morphism $q_{21} : N_2 \to P_1 \in \mathcal{Q}$ s.t. $q_{21} \circ n_2 = e_1 \circ h_{21}$ and (q_{21}, h_1) in $\mathcal{E}'$ (produce-forbid-conflict)

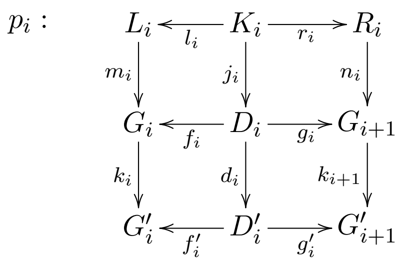


Now we prove that Definition 4.1 of critical pairs leads to completeness. This means, that each occuring conflict in a transformation system with NACs can be expressed by a critical pair i.e. by the same kind of conflict but in a minimal context. Therefore at first we need the following definition and lemma.

Definition 4.2 [extension diagram] An *extension diagram* is a diagram (1),

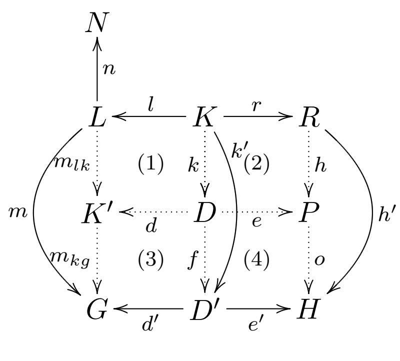


where, $k_0 : G_0 \to G'_0$ is a morphism, called extension morphism, and $t : G_0 \overset{*}{\Rightarrow} G_n$ and $t' : G'_0 \overset{*}{\Rightarrow} G'_n$ are transformations via the same rules $(p_0, \cdots, p_{n-1})$, and matches $(m_0, \cdots, m_{n-1})$ and extended matches $(k_0 \circ m_0, \cdots, k_{n-1} \circ m_{n-1})$, respectively, defined by the following DPO diagrams :



Remark 4.3 Since t and t' are transformations with NACs, the matches $(m_0, \cdots, m_{n-1})$ and extended matches $(k_0 \circ m_0, \cdots, k_{n-1} \circ m_{n-1})$ have to satisfy the NACs of the rules $(p_0, \cdots, p_{n-1})$.

Theorem 4.4 (Restriction Theorem with NACs) *Given a direct transformation $G \overset{p}{\Rightarrow} H$ with NACs via the rule $p : L \overset{l}{\leftarrow} K \overset{r}{\to} R$ and match $m : L \to G$ and given an object K' with two morphisms $L \overset{m_{lk}}{\to} K' \overset{m_{kg}}{\to} G$ s.t. $m = m_{kg} \circ m_{lk}$, with $m_{kg} \in \mathcal{M}'$, then there exists a direct transformation leading to the following extension diagram:*



Proof Given $G \overset{p}{\Rightarrow} H$ with NAC n as shown above. Since $d' \in \mathcal{M}$ we can take the pullback (3) of m_{kg} and d'. Since $m_{kg} \circ m_{lk} \circ l = m \circ l = d' \circ k'$ then there exists a morphism $k : K \to D$ with $k' = f \circ k$ and $d \circ k = m_{lk} \circ l$ because of the pullback property of (3). Because of the $\mathcal{M} - \mathcal{M}'$ pushout-pullback-decomposition property [3], $l \in \mathcal{M}$ and $m_{kg} \in \mathcal{M}'$, diagrams (1) and (3) are both pushouts. Now we can construct pushout (2) of $D \leftarrow K \to R$ because of $r \in \mathcal{M}$. Since

$e' \circ f \circ k = e' \circ k' = h' \circ r$ there exists a morphism $o : P \to H$ with $o \circ h = h'$ and $o \circ e = e' \circ f$ because of the pushout-property of (2). Because of the pushout-decomposition property also diagram (4) is a pushout.
It remains to show that m_{lk} satisfies the NACs of p. Suppose that m_{lk} doesn't satisfy some $NAC(n)$ of p, then there exists a morphism $q : N \to K' \in \mathcal{Q}$ s.t. $q \circ n = m_{lk}$, but this implies $m_{kg} \circ q \circ n = m_{kg} \circ m_{lk} = m$ with $q \in \mathcal{Q}$, $m_{kg} \in \mathcal{M}'$ and because of the composition property $m_{kg} \circ q \in \mathcal{Q}$, and this is a contradiction.□

Theorem 4.5 (Completeness of Critical Pairs with NACs) *For each pair of direct transformations* $H_1 \overset{(p_1,m_1')}{\Longleftarrow} G \overset{(p_2,m_2')}{\Longrightarrow} H_2$ *in conflict there is a critical pair* $P_1 \overset{p_1}{\Longleftarrow} K \overset{p_2}{\Longrightarrow} P_2$ *with extension diagrams (1) and (2) and* $m \in \mathcal{M}'$*.*

$$\begin{array}{ccccc} P_1 & \Longleftarrow & K & \Longrightarrow & P_2 \\ \downarrow & (1) & m\downarrow & (2) & \downarrow \\ H_1 & \Longleftarrow & G & \Longrightarrow & H_2 \end{array}$$

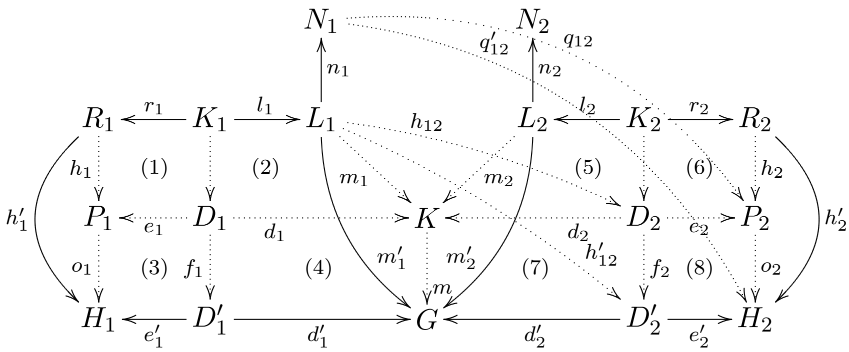
Proof According to Lemma 3.10 the following reasons are responsible for a pair of direct transformations $G \overset{(p_1,m_1')}{\Longrightarrow} H_1$ with NAC_{p_1} and $G \overset{(p_2,m_2')}{\Longrightarrow} H_2$ with NAC_{p_2} to be *in conflict*:

(i) (a) $\nexists h'_{12} : L_1 \to D'_2 : d'_2 \circ h'_{12} = m'_1$ (use-delete-conflict)
or
(b) there exists a unique $h'_{12} : L_1 \to D'_2 : d'_2 \circ h'_{12} = m'_1$, but $e'_2 \circ h'_{12} \not\models NAC_{p_1}$ (forbid-produce-conflict)
or

(ii) (a) $\nexists h'_{21} : L_2 \to D_1 : d'_1 \circ h'_{21} = m'_2$ (delete-use-conflict)
or
(b) there exists a unique $h'_{21} : L_2 \to D_1 : d'_1 \circ h'_{21} = m'_2$, but $e'_1 \circ h'_{21} \not\models NAC_{p_2}$ (produce-forbid-conflict)

It is possible, that (1.b) and (2.b) are both false. In this case, (1.a) or (2.a) have to be true which corresponds to the usual use-delete-conflict (resp. delete-use-conflict) and in [3] it is described how to embed a critical pair into this pair of direct transformations. In the other case, (1.b) or (2.b) are true. Let at first (1.b) be true. This means that there exists a unique $h'_{12} : L_1 \to D'_2 : d'_2 \circ h'_{12} = m'_1$, but $e'_2 \circ h'_{12} \not\models NAC_{p_1}$. Thus for one of the NACs $n_1 : L_1 \to N_1$ of p_1 there exists a morphism $q'_{12} : N_1 \to H_2 \in \mathcal{Q}$ such that $q'_{12} \circ n_1 = e'_2 \circ h'_{12}$. For each pair of morphisms with the same codomain, we have an $\mathcal{E}' - \mathcal{M}'$ pair factorization. Thus for $q'_{12} : N_1 \to H_2$ and $h'_2 : R_2 \to H_2$ we obtain an object P_2 and morphisms $h_2 : R_2 \to P_2$, $q_{12} : N_1 \to P_2$ and $o_2 : P_2 \to H_2$ with $(h_2, q_{12}) \in \mathcal{E}'$ and $o_2 \in \mathcal{M}'$ such that $o_2 \circ h_2 = h'_2$ and $o_2 \circ q_{12} = q'_{12}$. Because of Theorem 4.4 pushouts (5) - (8) can be constructed, if we consider the fact that also $H_2 \Rightarrow G$ is a direct transformation via p_2^{-1}. Since $o_2 \in \mathcal{M}'$ and (7) and (8) are pushouts along $\mathcal{M}$-morphisms also $f_2 \in \mathcal{M}'$ and $m \in \mathcal{M}'$. Because of the same argumentation as in Theorem 4.4, since m'_2 satisfies all the NACs of p_2 also m_2 satisfies them. Now we have the first half $K \Rightarrow P_2$ of the critical pair under construction. Now we can start constructing the second half of the critical pair. Let m_1 be the morphism $d_2 \circ h_{12}$, then the following

holds: $m \circ m_1 = m \circ d_2 \circ h_{12} = d'_2 \circ f_2 \circ h_{12} = d'_2 \circ h'_{12} = m'_1$. Because of Theorem 4.4 and $m \in \mathcal{M}'$ pushouts (1) - (4) can be constructed and because of the same argumentation as in Theorem 4.4 m_1 satisfies the NACs of p_1.
We still have to check if this critical pair is in forbid-produce-conflict. Since (8) is a pullback and $o_2 \circ q_{12} \circ n_1 = q'_{12} \circ n_1 = e'_2 \circ h'_{12}$ there exists a morphism $h_{12} : L_1 \to D_2$, with $e_2 \circ h_{12} = q_{12} \circ n_1$ and $f_2 \circ h_{12} = h'_{12}$. Because of the decomposition property, $q'_{12} = o_2 \circ q_{12} \in Q$ and $o_2 \in \mathcal{M}'$ we have $q_{12} \in \mathcal{Q}$. This means that $e_2 \circ h_{12}$ does not satisfy either the NAC $n_1 : L_1 \to N_1$.
Thus finally we obtain a critical pair according to Def. 4.1 of type (1.b) because we have h_{12} with $d_2 \circ h_{12} = m_1$. Moreover there is $q_{12} \in \mathcal{Q}$ with $(q_{12}, h_2) \in \mathcal{E}'$ and $e_2 \circ h_{12} = q_{12} \circ n_1$.



We can proceed analogously for the case of (2.b) being true leading to a critical pair of type (2.b) according to Def. 4.1. □

Fact 4.6 (necessary and sufficient condition for parallel independence) *Each pair of direct transformations $H_1 \Leftarrow G \Rightarrow H_2$ in an adhesive HLR system with NACs is parallel independent if and only if there are no critical pairs for this adhesive HLR system with NACs. An adhesive HLR system with NACs is locally confluent if there are no critical pairs for this adhesive HLR system with NACs.*

Proof

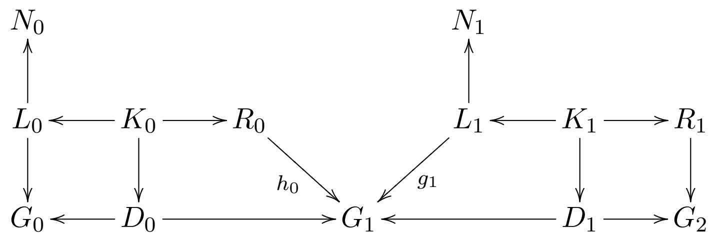
- Given an adhesive HLR system with NACs with an empty set of critical pairs and let $H_1 \Leftarrow G \Rightarrow H_2$ be a pair of direct transformations in conflict for this adhesive HLR system with NACs. This is a contradiction, since then there would exist a critical pair which can be embedded into this pair of direct transformations as shown in Theorem 4.5.
- Given an adhesive HLR system with NACs with only parallel independent pairs of direct transformations $H_1 \Leftarrow G \Rightarrow H_2$. Then the set of critical pairs has to be empty, otherwise a critical pair would be a pair of direct transformations in conflict.
- If each pair of direct transformations $H_1 \Leftarrow G \Rightarrow H_2$ in an adhesive HLR system with NACs is parallel independent then each pair is also locally confluent and using Theorem 3.3 in consequence this adhesive HLR system with NACs is locally confluent.

□

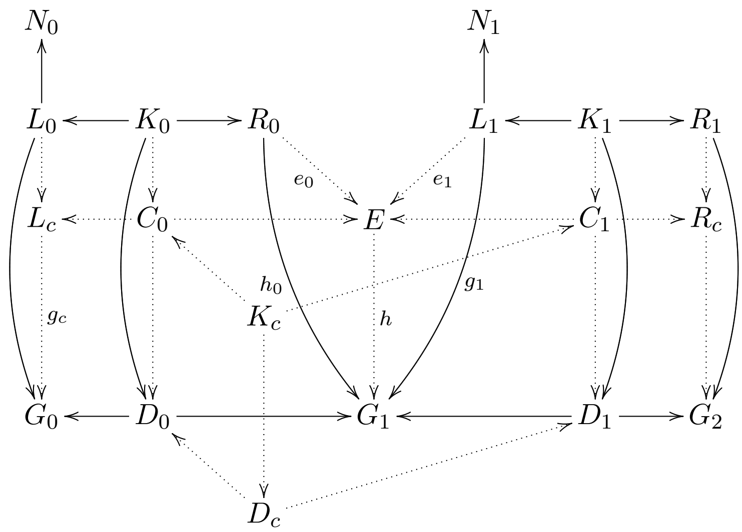
5 Concurrency in adhesive HLR Systems with NACs

Let t be a transformation via the rules $p_0, \cdots, p_{n-1}$ with NACs and matches $g_0, \cdots, g_{n-1}$. In general there will be dependencies between several direct transformations in this transformation sequence. Therefore it is not possible to apply the Parallelism Theorem in order to summarize the transformation sequence into one equivalent transformation step. It is possible though to formulate a Concurrency Theorem which expresses how to translate such a sequence into one equivalent transformation step anyway. Therefore we build on the notion of a concurrent rule of a transformation sequence without NACs as introduced in [3]. Moreover we have to translate all the NACs occuring in the rule sequence $p_0, \cdots, p_{n-1}$ backward into an equivalent set of NACs on the concurrent rule p_c of this rule sequence. This means, a set NAC_{p_c} should be found such that this set of NACs is equivalent to $NAC_{p_0}, \cdots, NAC_{p_{n-1}}$ for the transformation t. This section describes gradually how to obtain this concurrent NAC and generalizes then the Concurrency Theorem to transformations with NACs.

Let us consider at first a two-step transformation with NACs:



The goal is to translate all NACs on p_0 and p_1 into an equivalent set of NACs on the concurrent rule $p_c : L_c \leftarrow K_c \rightarrow R_c$ inducing as explained in Theorem 5.23 in [3] a concurrent transformation $G_0 \Rightarrow G_2$ via p_c and match g_c as shown in the following diagram:

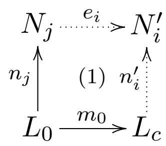


Consequently the two necessary steps are:

- Translate each set of NACs on L_0 into an equivalent set of NACs on L_c.
- Translate each set of NACs on L_1 into an equivalent set of NACs on L_c.

We can prove the first step as desribed in the following construction and Lemma.

Definition 5.1 [construction of NACs on L_c from NACs on L_0] Consider the following diagram:



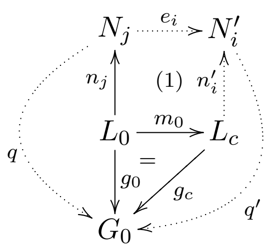
For each $NAC(n_j)$ on L_0 with $n_j : L_0 \to N_j$ and $m_0 : L_0 \to L_c$, let

$$D_{m_o}(NAC(n_j)) = \{NAC(n'_i) | i \in I, n'_i : L_c \to N'_i\}$$

where I and n'_i are constructed as follows: $i \in I$ if and only if (e_i, n'_i) with $e_i : N_j \to N'_i$ jointly epimorphic, $e_i \circ n_j = n'_i \circ m_0$ and $e_i \in \mathcal{Q}$.
For each set of NACs $NAC_{L_0} = \{NAC(n_j) | j \in J\}$ on L_0 the downward translation of NAC_{L_0} is then defined as:

$$D_{m_o}(NAC_{L_0}) = \cup_{j \in J} D_{m_o}(NAC(n_j))$$

Lemma 5.2 (equivalence of set of NACs on L_0 and set of NACs on L_c)
Given $g_c : L_c \to G_0$, $m_0 : L_0 \to L_c$ with NAC_{L_0} and $g_0 = g_c \circ m_0$ as in the following diagram:



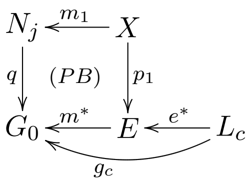
then the following holds :

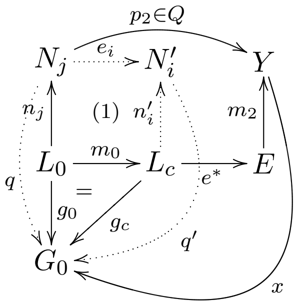
$$g_0 \models NAC_{L_0} \Leftrightarrow g_c \models D_{m_0}(NAC_{L_0})$$

Proof

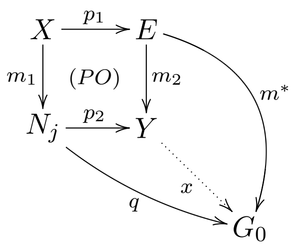
- ($\Rightarrow$) Let $g_c \not\models D_{m_0}(NAC(L_0)) = \cup_{j \in J} D_{m_0}(NAC(n_j))$ with $NAC_{L_0} = \{NAC(n_j) | j \in J\}$. Then for some $j \in J$ there is a NAC $n'_i : L_c \to N'_i \in D_{m_0}(NAC(n_j))$ and $e_i : N_j \to N'_i$ for which holds that $g_c \not\models NAC(n'_i)$, (e_i, n'_i) jointly epi, $e_i \in \mathcal{Q}$ and $e_i \circ n_j = n'_i \circ m_0$. Consequently there exists a morphism $q' : N'_i \to G_0 \in \mathcal{Q}$ such that $q' \circ n'_i = g_c$. Since $g_0 = g_c \circ m_0 = q' \circ n'_i \circ m_0 = q' \circ e_i \circ n_j$ there exists a morphism $q : N_j \to G_0$ defined by $q = q' \circ e_i$ s.t. $q \circ n_j = q' \circ e_i \circ n_j = g_0$. Because of the composition property for morphisms in Q we have $q \in \mathcal{Q}$ since $q' \in \mathcal{Q}$ and e_i in $\mathcal{Q}$. Hence $g_0 \not\models NAC(n_j) \Rightarrow g_0 \not\models NAC_{L_0}$.
- ($\Leftarrow$) Let $g_0 \not\models NAC_{L_0}$ with $NAC_{L_0} = \{NAC(n_j) | j \in J\}$. Then for some $j \in J$ a morphism $q : N_j \to G_0 \in Q$ exists such that $q \circ n_j = g_0$. Let (e^*, m^*) be an epi-$\mathcal{M}$-factorization of g_c. Construct X with $p_1 : X \to E$ and $m_1 : X \to N_j$ as

pullback of m^* and q.





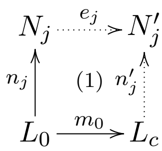
Then we have $m_1 \in \mathcal{M}$ and $p_1 \in \mathcal{Q}$, since $m^* \in \mathcal{M}$ and $q \in \mathcal{Q}$, PBs preserve $\mathcal{M}$ and PBs along $\mathcal{M}$ preserve $\mathcal{Q}$. Now construct Y with $m_2 : E \to Y$ and $p_2 : N_j \to Y$ as pushout of m_1 and p_1. Then we have $m_2 \in \mathcal{M}$, $p_2 \in \mathcal{Q}$, since $m_1 \in \mathcal{M}$, $p_1 \in \mathcal{Q}$, POs preserve $\mathcal{M}$ and POs along $\mathcal{M}$ preserve $\mathcal{Q}$. Because of the induced PB-PO property the induced morphism $x : Y \to G_0$ with $x \circ m_2 = m^*$ and $x \circ p_2 = q$ is a monomorphism in $\mathcal{Q}$.



It holds moreover that $p_2, m_2 \circ e^*$ jointly epimorphic because e^* epimorphic and p_2, m_2 jointly epimorphic. Summarizing we have the following equations: $x \circ m_2 \circ e^* \circ m_0 = m^* \circ e^* \circ m_0 = g_c \circ m_0 = g_0 = q \circ n_j = x \circ p_2 \circ n_j$ and since x mono we have $m_2 \circ e^* \circ m_0 = p_2 \circ n_j$. Since $m_2 \circ e^*$ and p_2 are jointly epimorphic, $p_2 \circ n_j = (m_2 \circ e^*) \circ m_0$ and $p_2 \in \mathcal{Q}$ we can conclude that $m_2 \circ e^* : L_c \to Y$ equals one of the morphisms $n'_i : L_c \to N'_i \in D_{m_0}(NAC(n_j))$. Moreover since $x \circ m_2 \circ e^* = m^* \circ e^* = g_c$ and $x \in \mathcal{Q}$ it holds that $g_c \not\models NAC(m_2 \circ e^*) = NAC(n'_i)$ and consequently $g_c \not\models NAC(n'_i) \Rightarrow g_c \not\models D_{m_0}(NAC(n_j)) \Rightarrow g_c \not\models D_{m_0}(NAC_{L_0})$. $\square$

Remark 5.3 It is possible to cancel the fact that Q is a set of special morphisms and thus generalize the definition of NAC-satisfiability. We should assume in this case though either that the NAC-morphism is in $\mathcal{M}$ or each match is in $\mathcal{M}$. For this case now we can use a different more simple downward translation of NACs onto the LHS of the concurrent rule expressed by the following definition and lemma.

Definition 5.4 [construction of NACs on L_c from NACs on L_0 for general $\mathcal{Q}$] Consider the following diagram:



For each $NAC(n_j)$ on L_0 with $n_j : L_0 \to N_j \in \mathcal{M}$ and $m_0 : L_0 \to L_c$, construct pushout (1) with $n'_j : L_c \to N'_j$ and $e_j : N_j \to N'_j$ then

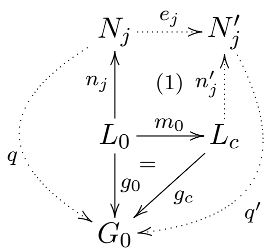
$$D_{m_o}(NAC(n_j)) = NAC(n'_j)$$

For each set of NACs $NAC_{L_0} = \{NAC(n_j) | j \in J\}$ on L_0 the downward translation of NAC_{L_0} is then defined as:

$$D_{m_o}(NAC_{L_0}) = \cup_{j \in J} D_{m_o}(NAC(n_j))$$

Lemma 5.5 (equivalence of NACs on L_c and NACs on L_0 for general $\mathcal{Q}$)

Given $g_c : L_c \to G_0$, $m_0 : L_0 \to L_c$ with NAC_{L_0} and $g_0 = g_c \circ m_0$ as in the following diagram:



then the following holds :

$$g_0 \models NAC_{L_0} \Leftrightarrow g_c \models D_{m_0}(NAC_{L_0})$$

Proof

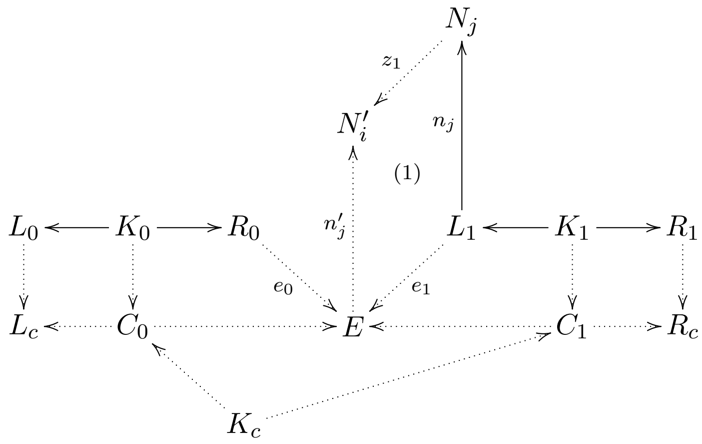
- $\Rightarrow$ Let $g_c \not\models D_{m_0}(NAC(L_0)) = \cup_{j \in J} D_{m_0}(NAC(n_j))$ with $NAC_{L_0} = \{NAC(n_j) | j \in J\}$. Then for some $j \in J$ there exists a morphism $q' : N'_j \to G_0$ with $q' \circ n'_j = g_c$. It follows that $q' \circ e_j \circ n_j = q' \circ n'_j \circ m_0 = g_c \circ m_0 = g_0$ and therefore $q' \circ e_j$ violates $NAC(n_j)$ for the match g_0 which is a contradiction.
- $\Leftarrow$ Let $g_0 \not\models NAC_{L_0}$ then for some $j \in J$ there exists a morphism $q : N_j \to G_0$ with $q \circ n_j = g_0$. Since (1) is a pushout and $q \circ n_j = g_0 = g_c \circ m_0$ there exists a morphism $q' : N'_j \to G_0$ with $q' \circ e_j = q$ and $q' \circ n'_j = g_c$. Because of the last equation $g_c \not\models D_{m_0}(NAC_{L_0})$ and this is a contradiction.

□

In Definition 2.9 and Lemma 2.11 it is explained how to construct an equivalent set of left NACs from a set of right NACs on a rule. Now we are ready to define a set of equivalent NACs on the LHS of the concurrent rule of a two-step transformation from the set of NACs on the LHS of the second rule of this transformation.

Definition 5.6 [construction of NACs on L_c from NACs on L_1] Given an E-dependency relation $(e_0, e_1) \in \mathcal{E}'$ for the rules p_0 and p_1 and $p_c = p_0 *_E p_1 : L_c \leftarrow$

$K_c \to R_c$ the E-concurrent rule of p_0 and p_1 as depicted in the following diagram:



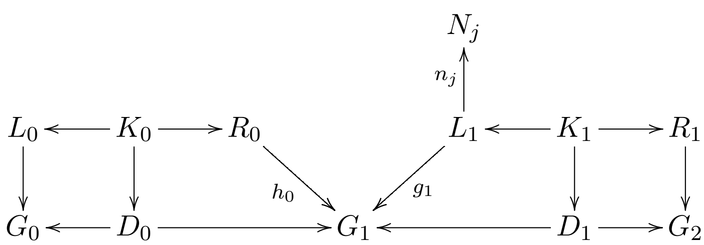
For each $NAC(n_j)$ on L_1 with $n_j : L_1 \to N_j$:

$$DL_{p_c}(NAC(n_j)) = L_p(D_{e_1}(NAC(n_j)))$$

with $p : L_c \leftarrow C_0 \to E$ and D_{e_1}, L_p according to Def. 5.1 and Def. 2.9.
For each set of NACs $NAC_{L_1} = \{NAC(n_j) | j \in J\}$ on L_1 the down- and leftward translation of NAC_{L_1} is defined as:

$$DL_{p_c}(NAC_{L_1}) = \cup_{j \in J} DL_{p_c}(NAC(n_j))$$

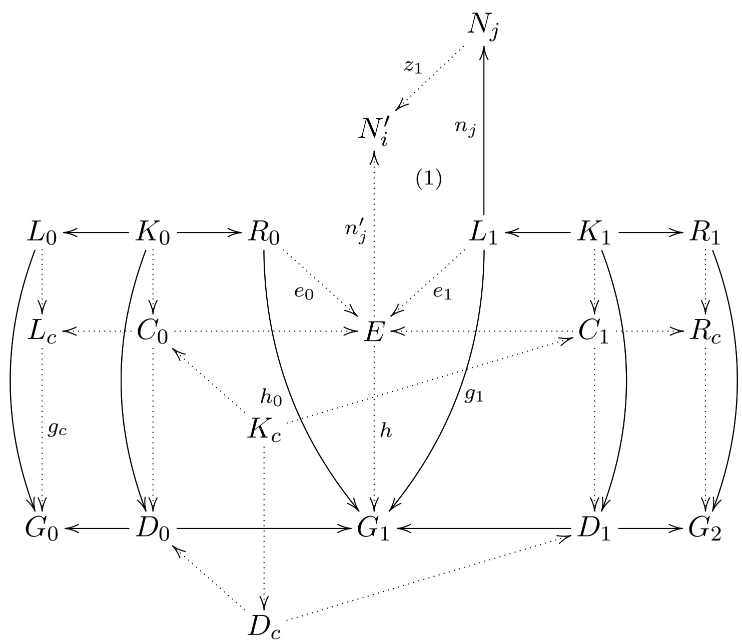
Lemma 5.7 (equivalence of NACs on rule p_1 and NACs on p_c) *Given a two-step E-related transformation via $p_0 : L_0 \leftarrow K_0 \to R_0$ and $p_1 : L_1 \leftarrow K_1 \to R_1$*



with g_c being the match from the LHS of the E-concurrent rule $p_c = p_0 *_E p_1$ into G_0 (as described in the synthesis construction of Theorem 5.23 in [3]) then the following holds:

$$g_1 \models NAC_{L_1} \Leftrightarrow g_c \models DL_{p_c}(NAC_{L_1}).$$

Proof Consider the following diagram:

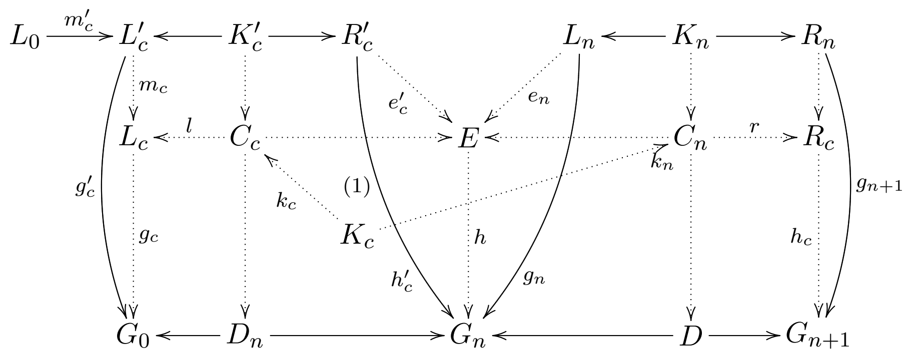


The E-concurrent rule $p_0 *_E p_1$ is the rule $p_c : L_c \leftarrow K_c \rightarrow R_c$, as described in Definition 5.21 in [3]. The derived span of the E-concurrent transformation $G_0 \overset{p_c,g_c}{\Rightarrow} G_2$ is $G_0 \leftarrow D_c \rightarrow G_2$. Because of Lemma 5.2 $g_1 \models NAC_{L_1} \Leftrightarrow h \models D_{e_1}(NAC_{L_1})$. Moreover because of Lemma 2.9 $g_c \models L_p(D_{e_1}(NAC_{L_1})) \Leftrightarrow h \models D_{e_1}(NAC_{L_1})$ with $p : L_c \leftarrow C_0 \rightarrow E$. Note that $L_p(D_{e_1}(NAC_{L_1})) = \cup_{j\in J} L_p(D_{e_1}(NAC_(n_j))) = \cup_{j\in J} DL_{p_c}(NAC(n_j))$. Consequently, it holds that $g_1 \models NAC_{L_1} \Leftrightarrow g_c \models DL_{p_c}(NAC_{L_1}))$. □

Definition 5.8 [concurrent rule with NAC, concurrent (co-, lhs-)match induced by $G_0 \overset{n+1}{\Longrightarrow} G_{n+1}$]

$n = 0$ For a direct transformation $G_0 \Rightarrow G_1$ via match $g_0 : L_0 \rightarrow G_0$, comatch $g_1 : R_1 \rightarrow G_1$ and rule $p_0 : L_0 \leftarrow K_0 \rightarrow R_0$ with NAC_{p_0} the *concurrent rule p_c with NAC* induced by $G_0 \Rightarrow G_1$ is defined by $p_c = p_0$ with $NAC_{p_c} = NAC_{p_0}$, the *concurrent comatch* h_c is defined by $h_c = g_1$, the *concurrent lhs-match* by $id : L_0 \rightarrow L_0$ and the *concurrent match* g_c by $g_c = g_0 : L_0 \rightarrow G_0$.

$n \geq 1$ Consider $p'_c : L'_c \leftarrow K'_c \rightarrow R'_c$ (resp. $g'_c : L'_c \rightarrow G_0$, $h'_c : R'_c \rightarrow G_n$,$m'_c : L_0 \rightarrow L'_c$), the concurrent rule with NACs (resp. concurrent match, comatch, lhs-match) induced by $G_0 \overset{n}{\Longrightarrow} G_n$. Let $((e'_c, e_n), h)$ be the $\mathcal{E}' - \mathcal{M}'$ pair factorization of the comatch h'_c and match g_n of $G_n \Rightarrow G_{n+1}$. According to Fact 5.29 in [3] PO-PB decomposition, PO composition and decomposition lead to the diagram below in which (1) is a pullback and all other squares are pushouts:



For a transformation sequence $G_0 \overset{n+1}{\Longrightarrow} G_{n+1}$ the *concurrent rule* p_c *with NACs* (resp. concurrent match, comatch, lhs-match) induced by $G_0 \overset{n+1}{\Longrightarrow} G_{n+1}$ is defined by $p_c = L_c \overset{l \circ k_c}{\leftarrow} K_c \overset{r \circ k_n}{\rightarrow} R_c$ ($g_c : L_c \to G_0$, $h_c : R_c \to G_{n+1}$, $m_c \circ m'_c : L_0 \to L_c$). Thereby NAC_{p_c} is defined by $NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$.

Theorem 5.9 (Concurrency Theorem with NACs) (i) Synthesis. *Given a transformation sequence* $t : G_0 \overset{*}{\Longrightarrow} G_{n+1}$ *via a sequence of rules* $p_0, p_1, \ldots, p_n$, *then there is a* synthesis construction *leading to a direct transformation* $G_0 \Rightarrow G_{n+1}$ *via the concurrent rule* $p_c : L_c \leftarrow K_c \rightarrow R_c$ *with* NAC_{p_c}, *match* $g_c : L_c \to G_0$ *and comatch* $h_c : R_c \to G_{n+1}$ *induced by* $t : G_0 \overset{*}{\Longrightarrow} G_{n+1}$.

(ii) Analysis. *Given a direct transformation* $G'_0 \Rightarrow G'_{n+1}$ *via the concurrent rule* $p_c : L_c \leftarrow K_c \rightarrow R_c$ *with* NAC_{p_c} *induced by* $t : G_0 \overset{*}{\Longrightarrow} G_{n+1}$ *via a sequence of rules* $p_0, p_1, \ldots, p_n$ *then there is an* analysis construction *leading to a transformation sequence* $t' : G'_0 \overset{*}{\Longrightarrow} G'_{n+1}$ *with NACs via* $p_0, p_1, \ldots, p_n$.

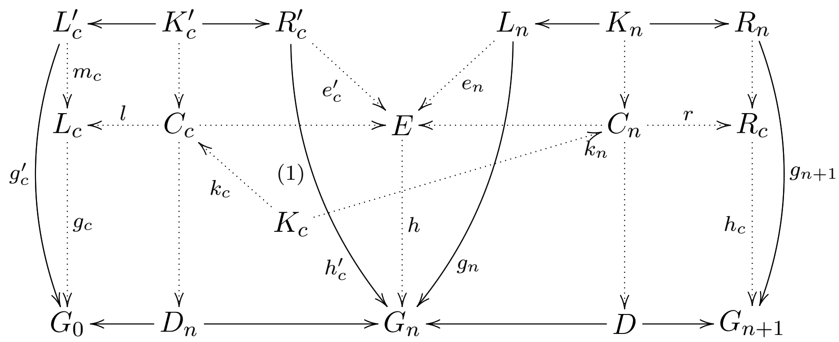
(iii) Bijective Correspondence. *The synthesis and analysis constructions are inverse to each other up to isomorphism.*

Proof We prove this theorem by induction over the number of transformation steps $n + 1$.

(i) *Synthesis.*

Basis. n=0. For a direct transformation $t : G_0 \overset{p_0, g_0}{\Rightarrow} G_1$ via match $g_0 : L_0 \to G_0$ and rule $p_0 : L_0 \leftarrow K_0 \rightarrow R_0$ with NAC_{p_0} the *concurrent rule* p_c *with NAC* induced by $G_0 \Rightarrow G_1$ is defined by $p_c = p_0$ with $NAC_{p_c} = NAC_{p_0}$ and the *concurrent match* g_c is defined by $g_c = g_0 : L_0 \to G_0$. Therefore the synthesis construction is equal to $G_0 \overset{p_c, g_c}{\Rightarrow} G_1$.

Induction Step. Consider $t : G_0 \overset{n}{\Longrightarrow} G_n \Rightarrow G_{n+1}$ via the rules $p_0, p_1 \ldots, p_n$. Let $p'_c : L'_c \leftarrow K'_c \rightarrow R'_c$ (resp. $g'_c : L'_c \to G_0$, $h'_c : R'_c \to G_n$), be the concurrent rule with NACs (resp. concurrent match, comatch) induced by $G_0 \overset{n}{\Longrightarrow} G_n$. Suppose that $G_0 \overset{p'_c, g'_c}{\Longrightarrow} G_n$ is a direct transformation with NAC leading to G_n. Let $((e'_c, e_n), h)$ be the $\mathcal{E}' - \mathcal{M}'$ pair factorization of the comatch h'_c and match g_n of $G_n \Rightarrow G_{n+1}$. PO-PB decomposition, PO composition and decomposition as described in Fact 5.29 in [3] lead to the diagram below in which (1) is a pullback and all other squares are pushouts:



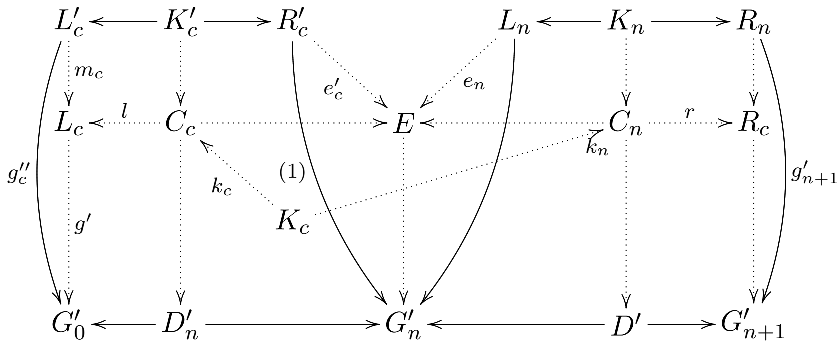
The *concurrent rule* p_c *with NACs* (resp. concurrent match, comatch) induced by $G_0 \overset{n+1}{\Longrightarrow} G_{n+1}$ is $p_c = L_c \overset{l \circ k_c}{\leftarrow} K_c \overset{r \circ k_n}{\rightarrow} R_c$ ($g_c : L_c \to G_0$, $h_c : R_c \to G_{n+1}$). Thereby NAC_{p_c} is $NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$. We should prove that $G_0 \overset{p_c, g_c}{\Longrightarrow} G_{n+1}$ is a valid direct transformation with NACs. At first an analogous synthesis construction to the one for two direct transformations without NACs in Theorem 5.23 in [3] can be done. Thus, in a second step we shall show that g_c satisfies NAC_{p_c} if g'_c satisfies $NAC_{p'_c}$ and g_n satisfies NAC_{p_n}. This follows because of Lemma 5.2, Lemma 5.7 and the fact that $G_0 \overset{p'_c, g'_c}{\Longrightarrow} G_n$ is a direct transformation via the rule p'_c with concurrent NAC $NAC_{p'_c}$.

(ii) *Analysis.*

Basis. n=0. For a direct transformation $G'_0 \Rightarrow G'_1$ via the concurrent rule $p_c = p_0$ with $NAC_{p_c} = NAC_{p_0}$ the analysis construction is equal to $G'_0 \Rightarrow G'_1$.

Induction Step. Given a direct transformation $G'_0 \Rightarrow G'_{n+1}$ via the concurrent rule $p_c : L_c \leftarrow K_c \rightarrow R_c$ with NAC_{p_c} induced by $t : G_0 \overset{*}{\Longrightarrow} G_{n+1}$ via a sequence of rules $p_0, p_1, \ldots, p_n$. The concurrent rule p_c induced by t can be interpreted as $p'_c *_{E_n} p_n$ in which the E_n-dependency relation between the rules is induced by the $\mathcal{E}' - \mathcal{M}'$ pair factorization of the comatch h'_c induced by $G_0 \overset{*}{\Longrightarrow} G_n$ and the match g_n of $G_n \to G_{n+1}$ as described in Def. 5.8. So we have a direct transformation $G'_0 \Rightarrow G'_{n+1}$ via $p_c = p'_c *_{E_n} p_n$ and because of the Analysis part of Theorem 5.23 in [3] there is an analysis construction leading to a transformation sequence without NACs $G'_0 \Rightarrow G'_n \Rightarrow G'_{n+1}$ via

p'_c and p_n and matches g''_c resp. g'_n.



We know by assumption that the match g' of $G'_0 \Rightarrow G'_{n+1}$ satisfies NAC_{p_c}. Since Lemma 5.2 and Lemma 5.7 hold in both directions, i.e. translate NACs in an equivalent way, we can conclude that $NAC_{p'_c}$ and NAC_{p_n} are satisfied by g''_c resp. g'_n. Therefore $G'_0 \Rightarrow G'_n \Rightarrow G'_{n+1}$ is a valid transformation sequence with NACs. Because of the induction hypothesis there exists an analysis construction $G'_0 \Rightarrow G'_1 \Rightarrow \dots G'_n$ via $p_0, p_1, \dots, p_{n-1}$ for $G'_0 \Rightarrow G'_n$ via p'_c. Thus we obtain a transformation sequence with NACs $G'_0 \Rightarrow G'_1 \Rightarrow \dots G'_{n+1}$ via $p_0, p_1, \dots, p_n$ for the direct transformation $G'_0 \Rightarrow G'_{n+1}$ via the concurrent rule $p_c : L_c \leftarrow K_c \rightarrow R_c$ with NAC_{p_c}.

(iii) *Bijective Correspondence.* The bijective correspondence follows from the fact that the $\mathcal{E}' - \mathcal{M}'$ pair factorization is unique, and pushout and pullback constructions are unique up to isomorphism.

□

6 Embedding and Confluence

In the last section we have defined the concurrent rule p_c with NACs (resp. match g_c) induced by a transformation t. We need this definition in order to define NAC-consistency which is an extra condition needed on top of boundary consistency to generalize the Embedding and Extension Theorem to transformations with NACs. Having generalized the notion of critical pairs, completeness, embedding and extension to transformations with NACs, it is now possible to formulate a sufficient condition on the critical pairs with NACs in order to obtain local confluence of an adhesive HLR system with NACs, i.e. to formulate the Critical Pair Lemma with NACs.

We start though with the definition of NAC-consistency for an extension morphism k_0 w.r.t. a transformation t. It expresses that the extended concurrent match induced by t should fullfill the concurrent NAC induced by t.

Definition 6.1 [NAC-consistency] A morphism $k_0 : G_0 \rightarrow G'_0$ is called NAC-consistent w.r.t. a transformation $t : G_0 \overset{*}{\Rightarrow} G_n$ if $k_0 \circ g_c \models NAC_{p_c}$ with NAC_{p_c} the *concurrent NAC* and g_c the *concurrent match induced by* t.

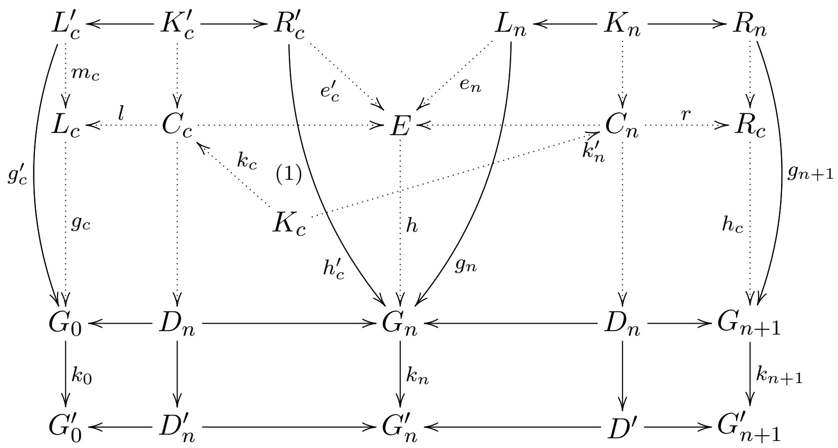
The Embedding Theorem for rules with NACs needs as extra condition on the extension morphism k_0 NAC-consistency. Note the following renaming in order to be able to distinguish better NAC-consistency from the consistency needed for the Embedding Theorem without NACs. In the following we speak about *boundary consistency* when we mean *consistency* as in Def. 6.12 of [3].

Theorem 6.2 (Embedding Theorem with NACs) *Given a transformation $t : G_0 \overset{n}{\Longrightarrow} G_n$ with NACs. If $k_0 : G_0 \to G'_0$ is boundary consistent and NAC-consistent w.r.t. t then there exists an extension diagram with NACs over t and k_0.*

Proof We prove this theorem by induction over the number of direct transformation steps n.

Basis. n=1. Consider a direct transformation $t : G_0 \overset{p_0,g_0}{\Rightarrow} G_1$ via match $g_0 : L_0 \to G_0$ and rule $p_0 : L_0 \leftarrow K_0 \to R_0$ with NAC_{p_0} and extension morphism $k_0 : G_0 \to G'_0$. Because of NAC-consistency $k_0 \circ g_0 \models NAC_{L_0}$. This means that the extension diagram over k_0 and t without NACs as described in Theorem 6.14 in [3] is also an extension diagram over k_0 and t with NACs.

Induction Step. Consider $t : G_0 \overset{n}{\Longrightarrow} G_n \overset{p_n,g_n}{\Rightarrow} G_{n+1}$ via the rules $p_0, p_1 \dots, p_n$. Let $p'_c : L'_c \leftarrow K'_c \to R'_c$ (resp. $g'_c : L'_c \to G_0$, $h'_c : R'_c \to G_n$) be the concurrent rule with NACs (resp. concurrent match, comatch) induced by $G_0 \overset{n}{\Longrightarrow} G_n$. The induction hypothesis says that there exists an extension diagram with NACs over $t' : G_0 \overset{n}{\Longrightarrow} G_n$ and $k_0 : G_0 \to G'_0$. This means in particular that $k_0 \circ g_0 \models NAC_{L_0}, k_1 \circ g_1 \models NAC_{L_1}, \dots, k_{n-1} \circ g_{n-1} \models NAC_{L_{n-1}}$, i.e. each extended match of the extension diagram satisfies the NACs on the corresponding rule. Moreover let $G'_0 \leftarrow D'_n \to G'_n$ be the derived span of the extension diagram $G'_0 \overset{n}{\Longrightarrow} G'_n$ over t and k_0. In the proof of Theorem 6.14 in [3] it is described how to obtain an extension diagram without NACs over $t : G_0 \overset{n}{\Longrightarrow} G_n \Rightarrow G_{n+1}$ and $k_0 : G_0 \to G'_0$. The same construction can be made since k_0 is boundary consistent. Now we still have to prove that the last extended match $k_n \circ g_n$ of the extension diagram without NACs satisfies the set of NACs on the last rule p_n of the transformation sequence. Let $((e'_c, e_n), h)$ be the $\mathcal{E}' - \mathcal{M}'$ pair factorization of the comatch h'_c and match g_n of $G_n \Rightarrow G_{n+1}$. PO-PB decomposition, PO composition and decomposition lead to the diagram below as described in Fact 5.29 in [3] in which (1) is a pullback and all other squares are pushouts:



The *concurrent rule p_c with NACs* (resp. concurrent match, comatch) induced by

$G_0 \overset{n+1}{\Longrightarrow} G_{n+1}$ is $p_c = L_c \overset{l \circ k_c}{\leftarrow} K_c \overset{r \circ k'_n}{\rightarrow} R_c$ $(g_c : L_c \to G_0,\ h_c : R_c \to G_{n+1})$. Thereby NAC_{p_c} is $NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$ and $G_0 \overset{p_c, g_c}{\Longrightarrow} G_n$. Because of NAC-consistency of k_0 we know that $k_0 \circ g_c \models NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$. Now because of Lemma 5.7 it follows that $k_n \circ g_n \models NAC_{L_n}$ and thus we have an extension diagram with NACs over t and k_0.

□

The following Extension Theorem with NACs describes the fact that boundary and NAC-consistency are not only sufficient, but also necessary conditions for the construction of extension diagrams for transformations with NACs.

Theorem 6.3 (Extension Theorem with NACs) *Given a transformation $t : G_0 \overset{n}{\Longrightarrow} G_n$ with NACs with a derived span $der(t) = (G_0 \overset{d_0}{\leftarrow} D_n \overset{d_n}{\rightarrow} G_n)$ and an extension diagram (1) as in the following picture:*

$$\begin{array}{ccccc} B & \xrightarrow{b_0} & G_0 & \overset{t}{\Longrightarrow}{}^{*} & G_n \\ \downarrow & (2) & \downarrow k_0 & (1) & \downarrow k_n \\ C & \longrightarrow & G'_0 & \overset{t'}{\Longrightarrow}{}^{*} & G'_n \end{array}$$

then

- *$k_0 : G_0 \to G'_0$ is boundary consistent w.r.t. t, with the morphism $b : B \to D_n$.*
- *$k_0 : G_0 \to G'_0$ is NAC-consistent w.r.t. t.*
- *Let p_c (resp. g_c) be the concurrent rule with NAC_{p_c} (resp. concurrent match) induced by t. There is a direct transformation $G'_0 \Rightarrow G'_n$ via $der(t)$ with $NAC_{der(t)} = D_{g_c}(NAC_{p_c})$ and match k_0 given by pushouts (3) and (4) with $h, k_n \in \mathcal{M}'$.*
- *There are initial pushouts (5) and (6) over $h \in \mathcal{M}'$ and $k_n \in \mathcal{M}'$, respectively, with the same boundary-context morphism $B \to C$.*

$$\begin{array}{ccccc} G_0 & \xleftarrow{d_0} & D_n & \xrightarrow{d_n} & G_n \\ \downarrow k_0 & (3) & \downarrow h & (4) & \downarrow k_n \\ G'_n & \longleftarrow & D'_n & \longrightarrow & G'_n \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{b} & D_n \\ \downarrow & (5) & \downarrow h \\ D'_n & \longrightarrow & D'_n \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{d_n \circ b} & D_n \\ \downarrow & (6) & \downarrow k_n \\ C & \longrightarrow & G'_n \end{array}$$

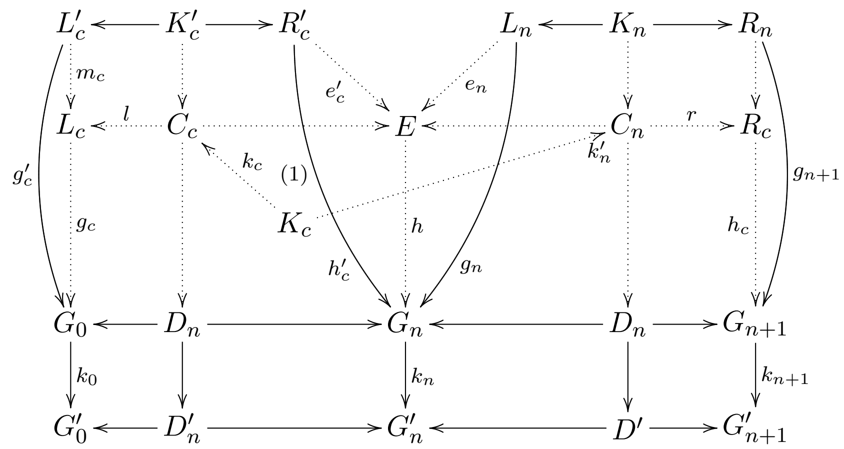
Proof

- See proof of item 1 in Theorem 6.16 in [3].
- We should prove that $k_0 \circ g_c$ with g_c the concurrent match induced by t satisfies NAC_{p_c} the concurrent NAC on the concurrent rule p_c induced by t. We prove this by induction over the number of direct transformation steps n.

  Basis. n=1. Consider the extension diagram over the direct transformation $t : G_0 \overset{p_0, g_0}{\Rightarrow} G_1$ (via match $g_0 : L_0 \to G_0$ and rule $p_0 : L_0 \leftarrow K_0 \to R_0$ with NAC_{p_0}) and extension morphism $k_0 : G_0 \to G'_0$. Because of Def. 4.2 $k_0 \circ g_0 \models NAC_{L_0}$ and therefore k_0 is NAC-consistent w.r.t. t.

  Induction Step. Consider the extension diagram over $t : G_0 \overset{n}{\Longrightarrow} G_n \overset{p_n, m_n}{\Rightarrow} G_{n+1}$ (via the rules $p_0, p_1 \ldots, p_n$ with NACs) and the extension morphism $k_0 : G_0 \to$

G'_0. Let $p'_c : L'_c \leftarrow K'_c \rightarrow R'_c$ (resp. $g'_c : L'_c \rightarrow G_0$, $h'_c : R'_c \rightarrow G_n$) be the concurrent rule with NACs (resp. concurrent match, comatch) induced by $G_0 \overset{n}{\Longrightarrow} G_n$. The induction hypothesis says that k_0 is NAC-consistent w.r.t. $t' : G_0 \overset{n}{\Longrightarrow} G_n$. This means in particular that $k_0 \circ g'_c$ satisfies $NAC_{p'_c}$. We should prove now that also $k_0 \circ g_c$ satisfies $NAC_{p_c} = DL_{p_c}(NAC_{L_n}) \cup D_{m_c}(NAC_{L'_c})$ the concurrent NAC on the concurrent rule p_c induced by t. Because of Lemma 5.2 and the induction hypothesis we have that $k_0 \circ g_c \models D_{m_c}(NAC_{L'_c}) = D_{m_c}(NAC_{p'_c})$. Because of Def. 4.2 $k_n \circ g_n \models NAC_{L_n}$ and because of Lemma 5.7 therefore $k_0 \circ g_c \models DL_{p_c}(NAC_{L_n})$.



- In item 2 in Theorem 6.16 in [3] it is proven that there is a direct transformation $G'_0 \Rightarrow G'_n$ without NACs via $der(t)$ and k_0 given by the pushouts (3) and (4) with $h, k_n \in \mathcal{M}'$. So we still have to prove that $k_0 \models NAC_{der(t)} = D_{g_c}(NAC_{p_c})$. This follows because of Lemma 5.2 and the fact that $k_0 \circ g_c \models NAC_{p_c}$ as proven in the former item.
- See proof of item 3 in Theorem 6.16 in [3].

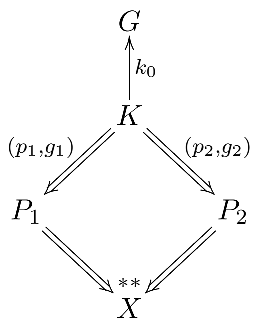
□

For the Critical Pair Lemma with NACs we need a stronger condition as in the case without NACs in order to obtain local conluence of the adhesive HLR system. In addition to strict confluence of the set of critical pairs we need also NAC-confluence. If a critical pair is strictly confluent via some transformations t_1 and t_2, we call t_1 and t_2 a *strict solution* of the critical pair. NAC-confluence of a critical pair expresses that the NAC-consistency of an extension morphism w.r.t. a strict solution of the critical pair follows from the NAC-consistency of the extension morphism w.r.t. the critical pair itself.

Definition 6.4 [strictly NAC-confluent] A critical pair $P_1 \overset{p_1,g_1}{\Leftarrow} K \overset{p_2,g_2}{\Rightarrow} P_2$ is *strictly NAC-confluent* if and only if

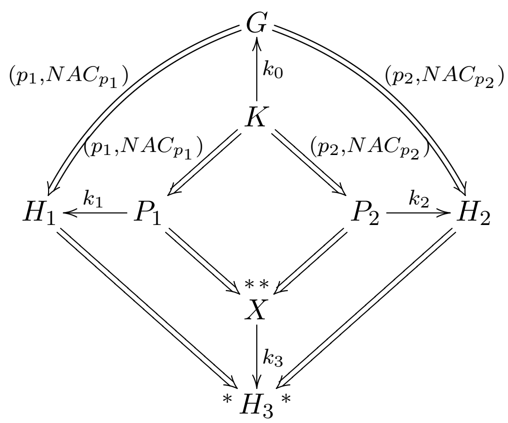
- it is *strictly confluent* via some transformations $t_1 : K \overset{p_1,g_1}{\Rightarrow} P_1 \Rightarrow^* X$ and $t_2 : K \overset{p_2,g_2}{\Rightarrow} P_2 \Rightarrow^* X$
- and it is *NAC-confluent for* t_1 *and* t_2, i.e. for every morphism $k_0 : K \rightarrow G \in \mathcal{M}'$ which is NAC-consistent w.r.t. $K \overset{p_1,g_1}{\Rightarrow} P_1$ and $K \overset{p_2,g_2}{\Rightarrow} P_2$ it follows that k_0 is

NAC-consistent w.r.t. t_1 and t_2.



Theorem 6.5 (Local Confluence Theorem - Critical Pair Lemma with NACs) *Given an adhesive HLR system with NACs, it is locally confluent if all its critical pairs are strictly NAC-confluent.*

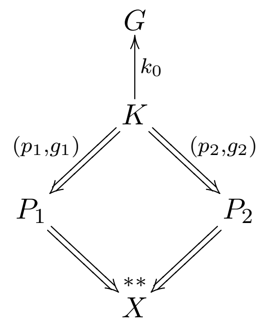
Proof If a pair of direct transformations is parallel independent then they are confluent because of the Local Church-Rosser Theorem. Suppose we have a pair of direct transformations $H_1 \Leftarrow G \Rightarrow H_2$ in conflict. Because of Theorem 4.5 a critical pair $P_1 \Leftarrow K \Rightarrow P_2$ exists, which can be embedded into this pair of direct transformations. We know that each critical pair is strictly NAC-confluent and consequently locally confluent. This means that an object X exists as in the following picture. We still have to prove though that $H_1 \Longrightarrow^* H_3 \Longleftarrow^* H_2$.



In the Critical Pair Lemma without NACs in [3] it is proven that the extension morphisms k_1 and k_2 are boundary consistent because of the fact that the critical pair $P_1 \Leftarrow K \Rightarrow P_2$ is not only locally confluent, but also strictly confluent. Therefore extension diagrams without NACs over $t_1 : K \Rightarrow P_1 \to^* X$ and $t_2 : K \Rightarrow P_2 \Rightarrow^* X$ can be constructed s.t. $G \Rightarrow H_1 \Rightarrow^* H_3$ and $G \Rightarrow H_2 \Rightarrow^* H_3$. The uniqueness of H_3 is proven in [3]. We should still prove that the extended matches in these extension diagrams satisfy the NACs of the rule sequences in t_1 and t_2. First we show that k_0 is boundary consistent w.r.t. $K \Rightarrow P_1$ (resp. $K \Rightarrow P_2$). This is because we have an extension diagram over k_0 and $K \Rightarrow P_1$ (resp. $K \Rightarrow P_2$) and Theorem 6.3. Moreover we have that $k_0 \circ g_1 \models NAC_{p_1}$ (resp. $k_0 \circ g_2 \models NAC_{p_2}$). This means that k_0 is also NAC-consistent w.r.t. the direct transformation $K \Rightarrow P_1$ (resp. $K \Rightarrow P_2$). Since we have that the critical pair $P_1 \Leftarrow K \Rightarrow P_2$ is strictly NAC-confluent it follows that

k_0 is NAC-consistent w.r.t t_1 and t_2. Since k_0 is NAC-consistent the Embedding Theorem for rules with NACs can be applied to k_0, t_1 and t_2 to conclude that the extended matches in the extension diagrams over k_0 and t_1 (resp. t_2) satisfy the NACs of the rule sequences in t_1 (resp. t_2). □

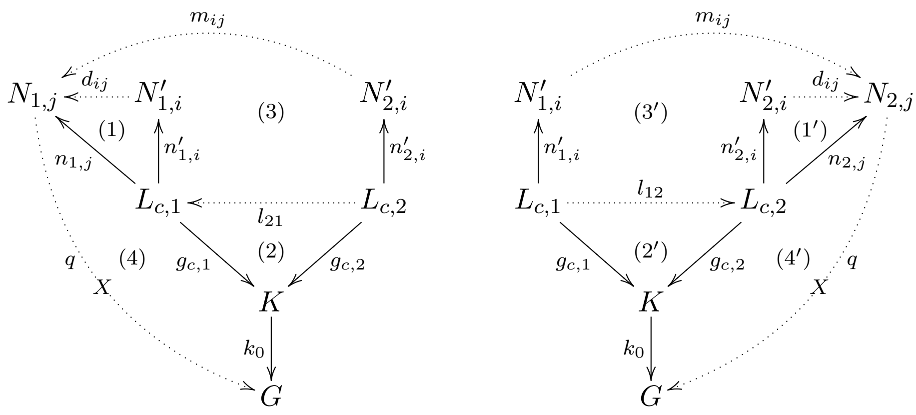
The following theorem formulates a sufficient condition for a critical pair $P_1 \overset{p_1}{\Leftarrow} K \overset{p_2}{\Rightarrow} P_2$ which is strictly confluent via some transformations $t_1 : K \overset{p_1,g_1}{\Rightarrow} P_1 \Rightarrow^* X$ and $t_2 : K \overset{p_2,g_2}{\Rightarrow} P_2 \Rightarrow^* X$ to be also NAC-confluent for t_1 and t_2. This means by definition that for every extension morphism $k_0 : K \to G \in \mathcal{M}'$ which is NAC-consistent w.r.t. $K \overset{p_1,g_1}{\Rightarrow} P_1$ and $K \overset{p_2,g_2}{\Rightarrow} P_2$ it should follow that k_0 is also NAC-consistent w.r.t. t_1 and t_2. In the theorem two different conditions on each single $NAC(n_{1,j})$ (resp. $NAC(n_{2,j})$) of the concurrent NAC induced by transformation t_1 (resp. t_2) are given which lead to NAC-confluence if one of them is satisfied. The first condition expresses that there exists a suitable NAC on p_1 (resp. p_2) which evokes the satisfaction of $NAC(n_{1,j})$ (resp. $NAC(n_{2,j})$). The second condition first asks for a suitable morphism between the lhs's of the concurrent rules induced by both transformations t_1 and t_2. Moreover it expresses that there exists a suitable NAC on p_2 (resp. p_1) which evokes the satisfaction of $NAC(n_{1,j})$ (resp. $NAC(n_{2,j})$).



Theorem 6.6 (Sufficient Condition for NAC-confluence) *Given a critical pair $P_1 \overset{p_1}{\Leftarrow} K \overset{p_2}{\Rightarrow} P_2$ which is strictly confluent via the transformations $t_1 : K \overset{p_1,g_1}{\Rightarrow} P_1 \Rightarrow^* X$ and $t_2 : K \overset{p_2,g_2}{\Rightarrow} P_2 \Rightarrow^* X$. Let $L_{c,1}$ (resp. $L_{c,2}$) be the left-hand side of the concurrent rule $p_{c,1}$ (resp. $p_{c,2}$) and $m_1 : L_1 \to L_{c,1}$ (resp. $m_2 : L_2 \to L_{c,2}$) the lhs-match induced by t_1 (resp. t_2). Then the critical pair $P_1 \overset{p_1}{\Leftarrow} K \overset{p_2}{\Rightarrow} P_2$ is also NAC-confluent for t_1 and t_2 and thus strictly NAC-confluent if one of the following conditions holds for each $NAC(n_{1,j}) : L_{c,1} \to N_{1,j}$ (resp. $NAC(n_{2,j}) : L_{c,2} \to N_{2,j}$) of the concurrent $NAC_{p_{c,1}}$ induced by t_1 (resp. $NAC_{p_{c,2}}$ induced by t_2)*

- *there exists a $NAC(n'_{1,i}) : L_{c,1} \to N'_{1,i}$ (resp. $NAC(n'_{2,i}) : L_{c,2} \to N'_{2,i}$) in $D_{m_1}(NAC_{p_1})$ (resp. $D_{m_2}(NAC_{p_2})$) and a morphism $d_{ij} \in \mathcal{Q} : N'_{1,i} \to N_{1,j}$ (resp. $d_{ij} \in \mathcal{Q} : N'_{2,i} \to N_{2,j}$) such that (1) (resp. (1')) commutes.*
- *there exists a morphism $l_{21} : L_{c,2} \to L_{c,1}$ (resp. $l_{12} : L_{c,1} \to L_{c,2}$) s.t. (2) (resp. (2')) commutes and in addition a $NAC(n'_{2,i}) : L_{c,2} \to N'_{2,i}$ (resp. $n'_{1,i} : L_{c,1} \to N'_{1,i}$) in $D_{m_2}(NAC_{p_2})$ (resp. $D_{m_1}(NAC_{p_1})$) with a morphism $m_{ij} : N'_{2,i} \to N_{1,j} \in \mathcal{Q}$ (resp. $m_{ij} : N'_{1,i} \to N_{2,j} \in \mathcal{M}$) s.t. $n_{1,j} \circ l_{21} = m_{ij} \circ n'_{2,i}$ (resp.*

$n_{2,j} \circ l_{12} = m_{ij} \circ n'_{1,i}$).



Proof We shall prove that k_0 is NAC-confluent for t_1 and t_2, i.e. each $k_0 : K \to G \in \mathcal{M}'$ which is NAC-consistent w.r.t. $K \overset{p_1}{\Rightarrow} P_1$ and $K \overset{p_2}{\Rightarrow} P_2$ is also NAC-consistent w.r.t. $t_1 : K \overset{p_1}{\Rightarrow} P_1 \Rightarrow^* X$ and $t_2 : K \overset{p_2}{\Rightarrow} P_2 \Rightarrow^* X$. Suppose that $k_0 \circ g_{c,1}$ does not satisfy the concurrent $NAC_{p_{c,1}}$ induced by t_1, i.e. k_0 is not NAC-consistent w.r.t. t_1. Then we have a morphism $q \in Q$ of a NAC-object $N_{1,j}$ of the concurrent $NAC_{p_{c,1}}$ into G s.t. triangle (4) commutes, i.e. $k_0 \circ g_{c,1} = q \circ n_{1,j}$. Now one of the following two reasonings can be made:

- Because of the existence of a morphism d_{ij} such that (1) commutes, we have that $k_0 \circ g_{c,1} = q \circ d_{ij} \circ n'_{1,i}$. Moreover $q \circ d_{ij}$ is a morphism in $\mathcal{Q}$ because $d_{ij}, q \in \mathcal{Q}$. This means that the extended match $k_0 \circ g_{c,1}$ does not satisfy $D_{m_1}(NAC_{p_1})$. Because of Lemma 5.2 now it follows that k_0 is not NAC-consistent w.r.t. $K \overset{p_1}{\Rightarrow} P_1$ and this is a contradiction.
- Because of the existence of $l_{21} : L_{c,2} \to L_{c,1}$ s.t. (2) commutes and $m_{ij} : N_{2,i} \to N_{1,j}$ such that $n_{1,j} \circ l_{21} = m_{ij} \circ n'_{2,i}$ the following equations hold: $q \circ m_{ij} \circ n'_{2,i} = q \circ n_{1,j} \circ l_{21} = k_0 \circ g_{c,1} \circ l_{21} = k_0 \circ g_{c,2}$. Now $q \circ m_{ij} \in \mathcal{Q}$ because of the composition property and thus $k_0 \circ g_{c,2}$ does not satisfy $D_{m_2}(NAC_{p_2})$. Because of Lemma 5.2 now it follows that k_0 is not NAC-consistent w.r.t. $K \overset{p_2}{\Rightarrow} P_2$ which is a contradiction.

Analogously we can prove that each $k_0 : K \to G$ which is NAC-consistent w.r.t. $K \Rightarrow P_1$ and $K \Rightarrow P_2$ is also NAC-consistent w.r.t. t_2. □

The following corollary follows directly from this theorem.

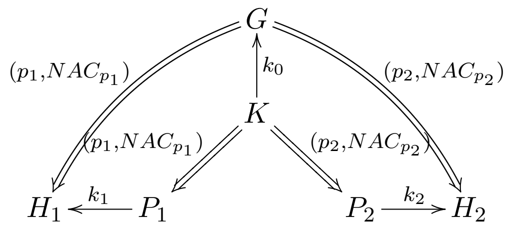
Corollary 6.7 *A critical pair $P_1 \overset{p_1}{\Leftarrow} K \overset{p_2}{\Rightarrow} P_2$ is strictly NAC-confluent if*

- *it is strictly confluent via the transformations $t_1 : K \overset{p_1,g_1}{\Rightarrow} P_1 \Rightarrow^* X$ (resp. $t_2 : K \overset{p_2,g_2}{\Rightarrow} P_2 \Rightarrow^* X$) and both $P_1 \Rightarrow^* X$ and $P_2 \Rightarrow^* X$ are transformation sequences without NACs.*

Proof In this case $NAC_{p_{c,1}} = D_{m_1}(NAC_{p_1})$ and therefore $d_{ij} = id$ for each single NAC in $NAC_{p_{c,1}}$ such that the first condition of Theorem 6.6 always holds. Analogously we can argue for the case $NAC_{p_{c,2}}$. □

In order to formulate not only a sufficient, but also necessary condition for local confluence of an adhesive HLR system with NACs we define the so-called set of extended critical pairs.

Definition 6.8 [extended critical pair] For each critical pair $P_1 \Leftarrow K \Rightarrow P_2$, which is not strictly NAC-confluent, an *extended critical pair* is a pair of direct transformations $H_1 \Leftarrow G \Rightarrow H_2$ as in the following figure, with $k_0 : K \rightarrow G \in \mathcal{M}'$ a NAC-consistent and boundary-consistent morphism w.r.t. $K \Rightarrow P_1$ and $K \Rightarrow P_2$:



Remark 6.9 If we take $k_0 = id$, then each critical pair which is not strictly NAC-confluent belongs to the set of its extended critical pairs.

Theorem 6.10 *An adhesive AHLR system with NACs is locally confluent if and only if all its extended critical pairs are locally confluent.*

Proof

- $\Rightarrow$ If one of the extended critical pairs is not locally confluent, then the system is not locally confluent, since an extended critical pair is nothing but a pair of direct transformations.
- $\Leftarrow$ Let $H_1 \Leftarrow G \Rightarrow H_2$ be a pair of direct transformations. If they are parallel independent, then they are locally confluent, because of the local Church-Rosser property. If they are not, then there exists a critical pair $P_1 \Leftarrow K \Rightarrow P_2$ because of Theorem 4.5 which can be embedded into this pair of direct transformations. If this critical pair is strictly NAC-confluent, then $H_1 \Leftarrow G \Rightarrow H_2$ is locally confluent as it is proven in Theorem 6.5. Suppose that the critical pair $P_1 \Leftarrow K \Rightarrow P_2$ is not strictly NAC-confluent. Because of Theorem 6.3 we have that $k_0 \in \mathcal{M}'$ is boundary- and NAC-consistent w.r.t. $K \Rightarrow P_1$ and $K \rightarrow P_2$. Then $H_1 \Leftarrow G \Rightarrow H_2$ is an extended critical pair by definition and all extended critical pairs are locally confluent by assumption.

□

7 Conclusion

In this paper the most significant results are given in order to extend the theory on Algebraic Graph Transformation in [3] to Transformations with Negative Application Conditions. Summarizing the most important ones: these are Local Church-Rosser Theorem, Parallelism Theorem, Completeness Theorem, Concurrency Theorem, Embedding and Extension Theorem, and Local Confluence Theorem. These results for transformations with NACs are formulated in the context of the Adhesive High-Level Replacement Framework introduced in [3] with an extra necessary morphism class $\mathcal{Q}$. The rather technical description of all theorems and proofs serves as a basis for further explanations and investigations on this subject together with illustrations of this new theory. Future work will be necessary on the applicability and refinement of all new results and on the development of efficient analysis algorithms for transformations with NACs.

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