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Construction and Properties of Adhesive and Weak Adhesive High-Level Replacement Categories

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Abstract As presented in Ehrig et al. (Fundamentals of Algebraic Graph Transformation, EATCS Monographs, Springer, 2006), adhesive high-level replacement (HLR) categories and systems are an adequate framework for several kinds of transformation systems based on the double pushout approach. Since (weak) adhesive HLR categories are closed under product, slice, coslice, comma and functor category constructions, it is possible to build new (weak) adhesive HLR categories from existing ones. But for the general results of transformation systems, as additional properties initial pushouts, binary coproducts compatible with a special morphism class $\mathcal{M}$ and a pair factorization are needed to obtain the full theory. In this paper, we analyze under which conditions these additional properties are preserved by the categorical constructions in order to avoid checking these properties explicitly.

Keywords Adhesive HLR categories · Initial pushouts · Coproducts · Pair factorization

Mathematics Subject Classifications (2000) 18A05 · 18A30 · 18A32 · 18C10 · 18D99

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1 Introduction

Adhesive high-level replacement categories and systems have been introduced recently in [6] as an integration of the concepts of high-level replacement (HLR) systems in [4, 5] – generalizing graph transformation systems – and of adhesive categories in [11, 12] – generalizing bisimulation congruences.

In [3], adhesive HLR systems are shown to be an adequate unifying framework for several interesting kinds of graph and net transformation systems. The main idea is to generalize the algebraic approach of graph transformations introduced in [2, 7] from graphs to high-level structures and to instantiate them with various kinds of graphs, Petri nets, algebraic specifications, and typed attributed graphs.

Adhesive and weak adhesive HLR categories are the basis of adhesive HLR systems. The concept of these categories is based on the existence and compatibility of suitable pushouts and pullbacks, which are essential for the so-called van Kampen (VK) squares, which have been introduced for adhesive categories in [11].

The idea of a VK square is that of a pushout which is stable under pullbacks, and, vice versa, that pullbacks are stable under combined pushouts and pullbacks. The name "van Kampen" is derived from the relationship between these squares and the Van Kampen theorem in topology (see [1]).

While adhesive categories are based on the class of all monomorphisms, adhesive and weak adhesive HLR categories $(\mathbf{C}, \mathcal{M})$ are based on a suitable subclass $\mathcal{M}$ of monomorphisms. This more flexible class $\mathcal{M}$ is essential for some important examples, such as typed attributed graphs, to become an adhesive HLR category. The concept of weak adhesive HLR categories is also important, because some other examples, such as place/transition nets and algebraic high-level nets (see [8]) satisfy only a weaker version of adhesive HLR categories. This weaker version, however, is still sufficient to obtain the basic main results for adhesive HLR systems in [3].

In [11], it was already observed how to extend the construction of adhesive categories from basic examples such as the category **Sets** of sets and functions to more complex examples such as the category **Graphs** of graphs and graph morphisms and the category $\mathbf{Graphs_{TG}}$ of graphs and graph morphisms typed over a type graph TG. In fact, it is claimed in [11] that adhesive categories are closed under product, slice, coslice and functor category constructions. This has been extended in [6] to adhesive HLR categories and in [3] it is shown in addition that adhesive and also weak adhesive HLR categories are closed under comma category constructions.

In [6] it is shown already that some results in the theory of adhesive HLR systems require some additional properties like finite coproducts compatible with $\mathcal{M}$, initial pushouts and a pair factorization. Especially the existence and construction of initial pushouts is nontrivial to be shown in several example categories. For this reason we study in this paper how far the constructions of (weak) adhesive HLR categories discussed above allow to preserve also these additional properties in order to avoid checking these properties explicitly.

This paper is organized as follows. In Section 2, we introduce (weak) adhesive HLR categories as presented in [3] and cite the important Construction theorem. In Section 3, we analyze under which conditions the additional properties are preserved under the categorical constructions mentioned above. Section 4 gives a conclusion and an overview of future work.

2 Adhesive and Weak Adhesive HLR Categories

In this section, we introduce the notion of adhesive and weak adhesive HLR categories, summarized as (weak) adhesive HLR categories, and present their closure under certain categorical constructions. The reader is assumed to be familiar with the basic notions of category theory, as presented in, e.g., [3, 9, 13]. For more motivation and examples we refer to [3].

The basic notion of (weak) adhesive HLR categories are van Kampen (VK) squares. The idea of a VK square is that of a pushout (PO) which is stable under pullbacks, and, vice versa, that pullbacks are stable under combined pushouts and pullbacks.

Definition 1 ([Weak] Van Kampen Square). A pushout (1) is a *van Kampen square* if, for any commutative cube (2) with (1) in the bottom and where the back faces are pullbacks, the following statement holds: the top face is a pushout iff the front faces are pullbacks:

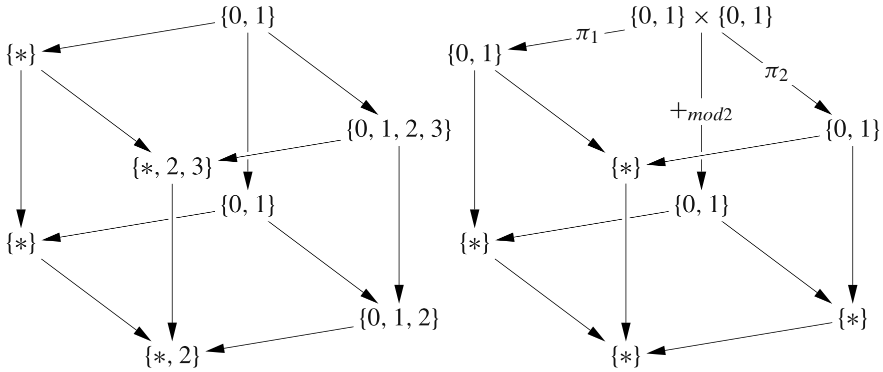
$$
\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "f"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "g"] \\
C \arrow[r, "n"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
C' \arrow[dr, "n'"] \arrow[dd, "c"'] & & A' \arrow[ll, "f'"'] \arrow[dr, "m'"] \arrow[dd, "a"] & \\
& D' \arrow[dd, "d"] & & B' \arrow[ll, "g'"'] \arrow[dd, "b"] \\
C \arrow[dr, "n"'] & & A \arrow[ll, "f"'] \arrow[dr, "m"] & \\
& D & & B \arrow[ll, "g"]
\end{tikzcd}
\quad (2)
$$

Given a morphism class $\mathcal{M}$, a pushout (1) with $m \in \mathcal{M}$ is a weak VK square if the above property holds for all commutative cubes with $f \in \mathcal{M}$ or $b, c, d \in \mathcal{M}$.

Remark Given a morphism class $\mathcal{M}$, a pushout (1) is a pushout along $\mathcal{M}$-morphisms if $m \in \mathcal{M}$ (or $f \in \mathcal{M}$). Analogously, a pullback (1) is a pullback along $\mathcal{M}$-morphisms if $n \in \mathcal{M}$ (or $g \in \mathcal{M}$).

Example 1 In the following diagram a VK square along an injective function in **Sets** is shown on the left-hand side. All morphisms are inclusions, except of 0 and 1 are mapped to $*$ and 2 and 3 to 2.

Arbitrary pushouts are stable under pullbacks in **Sets**. This means that one direction of the VK square property is also valid for arbitrary morphisms. However, the other direction is not necessarily valid. The cube on the right-hand side is such a counterexample, for arbitrary functions: all faces commute, the bottom and the top are pushouts, and the back faces are pullbacks. But, obviously, the front faces are not pullbacks, and therefore the pushout in the bottom fails to be a VK square.



Definition 2 ([Weak] Adhesive HLR Category). A category **C** with a morphism class $\mathcal{M}$ is called a (*weak*) *adhesive HLR category* if:

1. $\mathcal{M}$ is a class of monomorphisms closed under isomorphisms, composition ($f : A \to B \in \mathcal{M}, g : B \to C \in \mathcal{M} \Rightarrow g \circ f \in \mathcal{M}$), and decomposition ($g \circ f \in \mathcal{M}$, $g \in \mathcal{M} \Rightarrow f \in \mathcal{M}$).
2. **C** has pushouts and pullbacks along $\mathcal{M}$-morphisms, and $\mathcal{M}$-morphisms are closed under pushouts and pullbacks.
3. Pushouts in **C** along $\mathcal{M}$-morphisms are (weak) VK squares.

Example 2 The categories **Sets** and **Graphs** of sets and graphs, respectively, together with the class $\mathcal{M}$ of injective morphisms are adhesive HLR categories.

For the typing of graphs, a distinguished graph TG, called type graph, defines the available node and edge types. Then, a typed graph is a graph G together with a typing morphism $t : G \to TG$. Typed graph morphisms are graph morphisms that preserve the typing. The category $\mathbf{Graphs_{TG}}$ of typed graph together with the morphism class $\mathcal{M}$ of injective morphisms is an adhesive HLR category.

Also the category $\mathbf{AGraphs_{ATG}}$ of typed attributed graphs with the class $\mathcal{M}$ of injective morphisms with isomorphic data part is an adhesive HLR category. For typed attributed graphs, we have typing as well as attributes for nodes and edges, where the attribute values are specified by some algebra. Typed attributed graph morphisms map the graphs and the algebras such that both types and attribution is preserved.

The category **PTNets** of Petri nets with the morphism class $\mathcal{M}$ of injective morphisms is not an adhesive HLR category, but a weak adhesive HLR category, since only pullbacks along $\mathcal{M}$-morphisms are constructed componentwise, but not over general morphisms (see [8]).

(Weak) adhesive HLR categories are closed under product, slice, coslice, functor, and comma category constructions, where some of these notions are explained in the remark below. This means that we can construct new (weak) adhesive HLR categories from given ones.

Theorem 1 (Construction of [Weak] Adhesive HLR Categories). *If* $(\mathbf{C}, \mathcal{M}_1)$ *and* $(\mathbf{D}, \mathcal{M}_2)$ *are* (*weak*) *adhesive HLR categories, then we have the following results.*

1. *The product category* $(\mathbf{C} \times \mathbf{D}, \mathcal{M}_1 \times \mathcal{M}_2)$ *is a* (*weak*) *adhesive HLR category.*
2. *The slice category* $(\mathbf{C}\backslash X, \mathcal{M}_1 \cap \mathbf{C}\backslash X)$ *is a* (*weak*) *adhesive HLR category for any object* X *in* $\mathbf{C}$.
3. *The coslice category* $(X\backslash\mathbf{C}, \mathcal{M}_1 \cap X\backslash\mathbf{C})$ *is a* (*weak*) *adhesive HLR category for any object* X *in* $\mathbf{C}$.
4. *The comma category* $(ComCat(F, G; \mathcal{I}), \mathcal{M})$ *with* $\mathcal{M} = (\mathcal{M}_1 \times \mathcal{M}_2) \cap Mor_{ComCat(F,G;\mathcal{I})}$ *and with functors* $F : \mathbf{C} \to \mathbf{A}$, $G : \mathbf{D} \to \mathbf{A}$ *is a* (*weak*) *adhesive HLR category, if* F *preserves pushouts along* $\mathcal{M}_1$*-morphisms and* G *preserves pullbacks* (*along* $\mathcal{M}_2$*-morphisms*).
5. *The functor category* $([\mathbf{X}, \mathbf{C}], \mathcal{M}_1 - natural\ transformations)$ *is a* (*weak*) *adhesive HLR category for every category* $\mathbf{X}$.

Remark Given functors $F : \mathbf{C} \to \mathbf{A}$ and $G : \mathbf{D} \to \mathbf{A}$, and an index set $\mathcal{I}$, the objects of the comma category $(ComCat(F, G; \mathcal{I})$ are tuples $(C, D, (op_i)_{i\in\mathcal{I}})$ with $C \in \mathbf{C}$, $D \in \mathbf{D}$ and $op_i : F(C) \to G(D)$. A morphism $f : (C, D, op_i) \to (C', D', op'_i)$ consists of morphisms $f_C : C \to C'$ in $\mathbf{C}$ and $f_D : D \to D'$ in $\mathbf{D}$ such that $G(f_D) \circ op_i = op'_i \circ F(f_C)$ for all $i \in \mathcal{I}$.

$$\begin{tikzcd} F(C) \arrow[r, "op_i"] \arrow[d, "F(f_C)"] & G(D) \arrow[d, "G(f_D)"] \\ F(C') \arrow[r, "op'_i"] & G(D') \end{tikzcd}$$

In a functor category $[\mathbf{X}, \mathbf{C}]$, an $\mathcal{M}_1$-natural transformation is a natural transformation $t : F \to G$, where all morphisms $t_X : F(X) \to G(X)$ are in $\mathcal{M}_1$.

Proof See [3]. □

Example 3

1. The category **Graphs** × **Graphs** as the product category over **Graphs** together with injective morphisms is an adhesive HLR category (see item 5).
2. The category $\mathbf{Graphs_{TG}}$ as the slice category $\mathbf{Graphs}\backslash TG$ together with injective morphisms is an adhesive HLR category.
3. The category **PSets** of sets and partial functions is isomorphic to the coslice category $\{1\}\backslash\mathbf{Sets}$ and thus **PSets** together with injective total functions is an adhesive HLR category.
4. The category **PTNets** of Petri nets is isomorphic to the comma category $ComCat(F, G; \mathcal{I})$ with $F = Id : \mathbf{Sets} \to \mathbf{Sets}$, $G = \Box^{\oplus} : \mathbf{Sets} \to \mathbf{Sets}$ and $\mathcal{I} = \{pre, post\}$, where $A^{\oplus}$ is the free commutative monoid over A. Since F preserves pushouts and G preserves pullbacks along injective morphisms (but does not preserve general pullbacks), **PTNets** together with injective Petri net morphisms is a weak adhesive HLR category.

5. The category **Graphs** is isomorphic to the functor category $[\mathcal{S}_2, \mathbf{Sets}]$ with $\mathcal{S}_2 = \cdot \rightrightarrows \cdot$ and thus, together with injective graph morphisms, an adhesive HLR category.

To simplify the proofs, product, slice and coslice categories can be seen as special cases of comma categories. Hence, some results for these categories can be obtained from the corresponding results for comma categories.

Fact 1 (Product, Slice and Coslice Category as Comma Category). *For product, slice and coslice categories, we have the following isomorphic comma categories*:

1. $\mathbf{C} \times \mathbf{D} \cong ComCat(!_{\mathbf{C}} : \mathbf{C} \to \mathbf{1}, !_{\mathbf{D}} : \mathbf{D} \to \mathbf{1}, \varnothing)$,
2. $\mathbf{C}\backslash X \cong ComCat(id_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}, X : \mathbf{1} \to \mathbf{C}, \{1\})$ *and*
3. $X\backslash\mathbf{C} \cong ComCat(X : \mathbf{1} \to \mathbf{C}, id_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}, \{1\})$,

where $\mathbf{1}$ *is the final category,* $!_{\mathbf{C}} : \mathbf{C} \to \mathbf{1}$ *is the final morphism from* $\mathbf{C}$*, and* $X : \mathbf{1} \to \mathbf{C}$ *maps* $1 \in \mathbf{1}$ *to* $X \in \mathbf{C}$.

In the following, we need the fact below about pushouts in comma categories. It shows, that pushouts along $\mathcal{M}$-morphisms in comma categories based on (weak) adhesive HLR categories can be build componentwise from the underlying categories.

Fact 2 (Componentwise Pushouts in Comma Categories). *Given weak adhesive HLR categories* $(\mathbf{C}, \mathcal{M}_1)$ *and* $(\mathbf{D}, \mathcal{M}_2)$ *and functors* $F : \mathbf{C} \to \mathbf{A}$ *and* $G : \mathbf{D} \to \mathbf{A}$*, where* F *preserves pushouts along* $\mathcal{M}_1$*-morphisms, that lead to a comma category* $ComCat(F, G; \mathcal{I})$*. Then for objects* $A = (A_1, A_2, op_i^A)$*,* $B = (B_1, B_2, op_i^B)$*,* $C = (C_1, C_2, op_i^C) \in ComCat(F, G; \mathcal{I})$ *and morphisms* $a = (a_1, a_2) : A \to C$*,* $b = (b_1, b_2) : A \to B$ *with* $b \in \mathcal{M}_1 \times \mathcal{M}_2$ *we have*:

The diagram (1) *is a pushout in* $ComCat(F, G; \mathcal{I})$ *iff* $(1)_{\mathbf{C}}$ *and* $(1)_{\mathbf{D}}$ *are pushouts in* $\mathbf{C}$ *and* $\mathbf{D}$*, respectively, with* $f = (f_1, f_2)$ *and* $c = (c_1, c_2)$.

$$
\begin{tikzcd}
A_1 \arrow[r, "b_1"] \arrow[d, "a_1"'] \arrow[dr, phantom, "(1)_{\mathbf{C}}"] & B_1 \arrow[d, "f_1"] \\
C_1 \arrow[r, "c_1"'] & D_1
\end{tikzcd}
\quad
\begin{tikzcd}
A_2 \arrow[r, "b_2"] \arrow[d, "a_2"'] \arrow[dr, phantom, "(1)_{\mathbf{D}}"] & B_2 \arrow[d, "f_2"] \\
C_2 \arrow[r, "c_2"'] & D_2
\end{tikzcd}
\quad
\begin{tikzcd}
A \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & B \arrow[d, "f"] \\
C \arrow[r, "c"'] & D
\end{tikzcd}
$$

Proof "$\Leftarrow$" Given the morphisms a and b in (1), and the pushouts $(1)_{\mathbf{C}}$ and $(1)_{\mathbf{D}}$ in $\mathbf{C}$ and $\mathbf{D}$, respectively. We have to show that (1) is a pushout in $ComCat(F, G; \mathcal{I})$.

Since F preserves pushouts along $\mathcal{M}_1$-morphisms, with $b_1 \in \mathcal{M}_1$ the diagram (2) is a pushout. Then $D = (D_1, D_2, op_i^D)$ is an object in $ComCat(F, G; \mathcal{I})$, where, for $i \in \mathcal{I}$, op_i^D is the by (2) and $G(c_2) \circ op_i^C \circ F(a_1) = G(c_2) \circ G(a_2) \circ op_i^B = G(f_2) \circ G(b_2) \circ op_i^B = G(f_2) \circ op_i^A \circ F(b_1)$ induced morphism with $op_i^D \circ F(c_1) = G(c_2) \circ op_i^C$ and $op_i^D \circ F(f_1) = G(f_2) \circ op_i^A$. Therefore $c = (c_1, c_2)$ and $f = (f_1, f_2)$ are morphisms in $ComCat(F, G; \mathcal{I})$ such that (1) commutes.

$$\begin{tikzcd}
G(A_2) \arrow[rrr, "G(b_2)"] \arrow[ddd, "G(a_2)"'] & & & G(B_2) \arrow[ddd, "G(f_2)"] \arrow[dddddrr, bend left, "G(k_2)"] & & \\
& F(A_1) \arrow[ul, "op_i^B"] \arrow[r, "F(b_1)"] \arrow[d, "F(a_1)"'] \arrow[dr, phantom, "(2)"] & F(B_1) \arrow[ur, "op_i^A"'] \arrow[d, "F(f_1)"] \arrow[ddrr, "F(k_1)"] & & & \\
& F(C_1) \arrow[dl, "op_i^C"'] \arrow[r, "F(c_1)"] \arrow[ddrrr, bend right, "F(h_1)"] & F(D_1) \arrow[dr, "op_i^D"] \arrow[ddrr, "F(x_1)"] & & & \\
G(C_2) \arrow[rrr, "G(c_2)"] \arrow[ddrrrrr, bend right, "G(h_2)"'] & & & G(D_2) \arrow[ddrr, "G(x_2)"] & & \\
& & & & F(X_1) \arrow[dr, "op_i^X"] & \\
& & & & & G(X_2)
\end{tikzcd}$$

It remains to show that (1) is a pushout. Given an object $X = (X_1, X_2, op_i^X)$ and morphisms $h = (h_1, h_2) : C \to X$ and $k = (k_1, k_2) : A \to X$ in $ComCat(F, G; \mathcal{I})$ such that $h \circ a = k \circ b$. From pushouts $(1)_{\mathbf{C}}$ and $(1)_{\mathbf{D}}$ we obtain unique morphisms $x_1 : D_1 \to X_1$ and $x_2 : D_2 \to X_2$ such that $x_i \circ c_i = h_i$ and $x_i \circ f_i = k_i$ for $i = 1, 2$. Since (2) is a pushout, from $G(x_2) \circ op_i^D \circ F(c_1) = G(x_2) \circ G(c_2) \circ op_i^C = G(h_2) \circ op_i^C = op_i^X \circ F(h_1) = op_i^X \circ F(x_1) \circ F(c_1)$ and $G(x_2) \circ op_i^D \circ F(f_1) = G(x_2) \circ G(f_2) \circ op_i^A = G(k_2) \circ op_i^A = op_i^X \circ F(k_1) = op_i^X \circ F(x_1) \circ F(f_1)$ it follows that $G(x_2) \circ op_i^D = op_i^X \circ F(x_1)$. Therefore $x = (x_1, x_2) \in ComCat(F, G; \mathcal{I})$, and x is unique with respect to $x \circ c = h$ and $x \circ f = k$.

"$\Rightarrow$" Given the pushout (1) in $ComCat(F, G; \mathcal{I})$, we have to show that $(1)_{\mathbf{C}}$ and $(1)_{\mathbf{D}}$ are pushouts in $\mathbf{C}$ and $\mathbf{D}$, respectively.

Since $(\mathbf{C}, \mathcal{M}_1)$ and $(\mathbf{D}, \mathcal{M}_2)$ are weak adhesive HLR categories there exist pushouts $(1')_{\mathbf{C}}$ and $(1')_{\mathbf{D}}$ over a_1 and $b_1 \in \mathcal{M}_1$ in $\mathbf{C}$ and over a_2 and $b_2 \in \mathcal{M}_2$ in $\mathbf{D}$, respectively.

$$\begin{tikzcd}
B_1 \arrow[r, "b_1"] \arrow[d, "a_1"'] \arrow[dr, phantom, "(1')_{\mathbf{C}}"] & A_1 \arrow[d, "g_1"] & B_2 \arrow[r, "b_2"] \arrow[d, "a_2"'] \arrow[dr, phantom, "(1')_{\mathbf{D}}"] & A_2 \arrow[d, "g_2"] & B \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1')"] & A \arrow[d, "g"] \\
C_1 \arrow[r, "e_1"] & E_1 & C_2 \arrow[r, "e_2"] & E_2 & C \arrow[r, "e"] & E
\end{tikzcd}$$

Therefore (using "$\Leftarrow$") there is a corresponding pushout $(1')$ in $ComCat(F, G; \mathcal{I})$ over a and b with $E = (E_1, E_2, op_i^E)$, $e = (e_1, e_2)$ and $g = (g_1, g_2)$. Since pushouts are unique up to isomorphism it follows that $E \tilde{=} D$, which means $E_1 \tilde{=} D_1$ and $E_2 \tilde{=} D_2$ and therefore $(1)_{\mathbf{C}}$ and $(1)_{\mathbf{D}}$ are pushouts in $\mathbf{C}$ and $\mathbf{D}$, respectively. □

From Facts 1 and 2 we obtain immediately the following Fact 3.

Fact 3 (Componentwise POs in Product, Slice, Coslice Categories). *Since* $\mathbf{C} \times \mathbf{D} \tilde{=} ComCat(!_{\mathbf{C}} : \mathbf{C} \to \mathbf{1}, !_{\mathbf{D}} : \mathbf{D} \to \mathbf{1}, \varnothing)$, $\mathbf{C}\backslash X \tilde{=} ComCat(id_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}, X : \mathbf{1} \to \mathbf{C}, \{1\})$

and $X \backslash \mathbf{C} \tilde{=} ComCat(X : \mathbf{1} \to \mathbf{C}, id_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}, \{1\})$ *and the functors* $!_{\mathbf{C}}$, $id_{\mathbf{C}}$ *and* X *preserve pushouts, it follows that also in product, slice and coslice categories pushouts along* $\mathcal{M}$*-morphisms are constructed componentwise.*

3 Additional Properties and Their Preservation

There are several important results in the theory of adhesive HLR systems in [3] where we need (weak) adhesive HLR categories which satisfy some additional properties. In this section we analyze, under which conditions these properties are preserved by the categorical constructions discussed in Section 2. Section 3.1 handles initial pushouts, Section 3.2 handles finite coproducts compatible with $\mathcal{M}$ and in Section 3.3, the preservation of an $\mathcal{E}'$-$\mathcal{M}'$ pair factorization is described.

These properties are needed in the theory of adhesive HLR systems as discussed in Chapters 5 and 6 of [3], where coproducts are used for parallel productions, initial pushouts for the extension of transformations and local confluence, and pair factorizations for concurrent productions and critical pairs.

For the existence of initial pushouts, we need a special morphism class $\mathcal{M}'$, since in some categories initial pushouts over general morphisms do not exist or have a more complicated structure that cannot be derived from the categorical constructions. This morphism class $\mathcal{M}'$ is also used for the $\mathcal{E}'$-$\mathcal{M}'$ pair factorization, where $\mathcal{E}'$ is a class of morphism pairs with the same codomain. For the local confluence theorem in [3], we need an $\mathcal{E}'$-$\mathcal{M}'$ pair factorization with initial pushouts over $\mathcal{M}'$-morphisms.

3.1 Initial Pushouts

An initial pushout formalizes the construction of the boundary and context of a morphism. For a morphism $f : A \to A'$, we want to construct a boundary $b : B \to A$, a boundary object B, and a context object C, leading to a minimal pushout satisfying an initiality property. Roughly speaking, A' is the gluing of A and the context object C along the boundary object B.

Definition 3 (Initial Pushout). Given a morphism $f : A \to A'$ in a (weak) adhesive HLR category, a morphism $b : B \to A$ with $b \in \mathcal{M}$ is called the *boundary* over f if there is a pushout complement of f and b such that (1) is a pushout which is *initial* over f. Initiality of (1) over f means, that for every pushout (2) with $b' \in \mathcal{M}$ there exist unique morphisms $b^* : B \to D$ and $c^* : C \to E$ with $b^*, c^* \in \mathcal{M}$ such that $b' \circ b^* = b$, $c' \circ c^* = c$ and (3) is a pushout.

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d] \arrow[dr, phantom, "(1)"] & A \arrow[d, "f"] \\
C \arrow[r, "c"] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "b^*"] \arrow[rr, bend left, "b"] \arrow[d] \arrow[dr, phantom, "(3)"] & D \arrow[r, "b'"] \arrow[d] \arrow[dr, phantom, "(2)"] & A \arrow[d, "f"] \\
C \arrow[r, "c^*"] \arrow[rr, bend right, "c"'] & E \arrow[r, "c'"] & A'
\end{tikzcd}
$$

Remark If an initial pushouts exists only over $f \in \mathcal{M}'$ for a special class $\mathcal{M}'$ of morphisms, we say that $(\mathbf{C}, \mathcal{M})$ has initial pushouts over $\mathcal{M}'$.

Example 4 Initial pushouts exist in **Sets**, **Graphs**, $\mathbf{Graphs_{TG}}$ and $\mathbf{AGraphs_{ATG}}$ (see [3]). For **Graphs** and $\mathbf{Graphs_{TG}}$, they can be constructed by Theorem 2.

Theorem 2 (Preservation of Initial Pushouts). *Given (weak) adhesive HLR categories* $(\mathbf{C}, \mathcal{M}_1)$ *and* $(\mathbf{D}, \mathcal{M}_2)$ *with initial pushouts over* $\mathcal{M}'_1$*- and* $\mathcal{M}'_2$*-morphisms, respectively, for some morphisms classes* $\mathcal{M}'_1$ *in* $\mathbf{C}$ *and* $\mathcal{M}'_2$ *in* $\mathbf{D}$*. Then we have the following results.*

1. $\mathbf{C} \times \mathbf{D}$ *has initial pushouts over* $\mathcal{M}'_1 \times \mathcal{M}'_2$*-morphisms.*
2. $\mathbf{C}\backslash X$ *has initial pushouts over* $\mathcal{M}'_1 \cap \mathbf{C}\backslash X$*-morphisms.*
3. *For* $f : A \to D \in \mathcal{M}'_1 \cap X\backslash\mathbf{C}$ *with* $a' : X \to A$ *the initial pushout exists, if*
   - *the initial pushout over* f *in* $\mathbf{C}$ *can be extended to a valid square in* $X\backslash\mathbf{C}$ *or*
   - $a' : X \to A \in \mathcal{M}_1$ *and the pushout complement of* a' *and* f *in* $\mathbf{C}$ *exists.*
4. *If* F *preserves pushouts along* $\mathcal{M}_1$*-morphisms and* $G(\mathcal{M}_2) \subseteq Isos$*, then* $ComCat(F, G; \mathcal{I})$ *has initial pushouts over* $\mathcal{M}'_1 \times \mathcal{M}'_2$*- morphisms.*
5. *If* $\mathbf{C}$ *has intersections of* $\mathcal{M}_1$*-subobjects (see remark below) then* $[\mathbf{X}, \mathbf{C}]$ *has initial pushouts over* $\mathcal{M}'_1$*-natural transformations.*

Remark A category $\mathbf{C}$ has intersections of $\mathcal{M}_1$-subobjects, if it has the following kind of limits compatible with $\mathcal{M}_1$: Given $c_i : C_i \to D \in \mathcal{M}_1$ with $i \in I$ for some index set I, then the corresponding diagram has a limit $(C, (c'_i : C \to C_i)_{i \in I}, c : C \to D)$ in $\mathbf{C}$ and we have that $c'_i \in \mathcal{M}_1$ for all $i \in I$ and $c \in \mathcal{M}_1$.

$$\begin{tikzcd} & C_i \arrow[dr, "c_i"] & \\ C \arrow[ur, "c'_i"] \arrow[rr, "c"] \arrow[dr, "c'_j"'] & & D \\ & C_j \arrow[ur, "c_j"'] & \end{tikzcd} \qquad (i, j \in I)$$

Proof

1. Since $\mathbf{C} \times \mathbf{D} \stackrel{\sim}{=} ComCat(!_\mathbf{C} : \mathbf{C} \to \mathbf{1}, !_\mathbf{D} : \mathbf{D} \to \mathbf{1}, \varnothing)$, $!_\mathbf{C}$ preserves pushouts and $!_\mathbf{D}(\mathcal{M}_2) \subseteq \{id_1\} = Isos$ this follows from item 4. Then the initial pushout (3) of a morphism $(f_1, f_2) : (A_1, A_2) \to (D_1, D_2) \in \mathcal{M}'_1 \times \mathcal{M}'_2$ is the product of the initial pushouts (1) over f_1 in $\mathbf{C}$ and (2) over f_2 in $\mathbf{D}$.

$$\begin{tikzcd} B_1 \arrow[r, "b_1"] \arrow[d, "a_1"'] \arrow[dr, phantom, "(1)"] & A_1 \arrow[d, "f_1"] \\ C_1 \arrow[r, "c_1"] & D_1 \end{tikzcd} \quad \begin{tikzcd} B_2 \arrow[r, "b_2"] \arrow[d, "a_2"'] \arrow[dr, phantom, "(2)"] & A_2 \arrow[d, "f_2"] \\ C_2 \arrow[r, "c_2"] & D_2 \end{tikzcd} \quad \begin{tikzcd} (B_1, B_2) \arrow[r, "(b_1,b_2)"] \arrow[d, "(a_1,a_2)"'] \arrow[dr, phantom, "(3)"] & (A_1, A_2) \arrow[d, "(f_1,f_2)"] \\ (C_1, C_2) \arrow[r, "(c_1,c_2)"] & (D_1, D_2) \end{tikzcd}$$

2. Since $\mathbf{C}\backslash X \stackrel{\sim}{=} ComCat(id_\mathbf{C} : \mathbf{C} \to \mathbf{C}, X : \mathbf{1} \to \mathbf{C}, \{1\})$, $id_\mathbf{C}$ preserves pushouts and $X(\mathcal{M}_2) = X(\{id_1\}) = \{id_X\} \subseteq Isos$ this follows from item 4. Then the initial

pushout (2) over $f : (A, a') \to (D, d') \in \mathcal{M}'_1$ in $\mathbf{C}\backslash X$ is given by the initial pushout (1) over f in $\mathbf{C}$, with $b' = a' \circ b$ and $c' = d' \circ c$.

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "f"] \\ C \arrow[r, "c"] & D \end{tikzcd} \qquad \begin{tikzcd} (B, b') \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(2)"] & (A, a') \arrow[d, "f"] \\ (C, c') \arrow[r, "c"] & (D, d') \end{tikzcd}$$

3. Given objects (A, a'), (D, d') and a morphism $f : A \to D$ in $X\backslash\mathbf{C}$ with $f \in \mathcal{M}'_1$. Then the initial pushout (1) over f in $\mathbf{C}$ exists by assumption. The remark of Fact 2 implies that, for any pushout (2) in $X\backslash\mathbf{C}$ with $d, e \in \mathcal{M}_1$, the diagram (3) is a pushout in $\mathbf{C}$. Since (1) is an initial pushout in $\mathbf{C}$ there exist unique morphisms $b^* : B \to E$ and $c^* : C \to F$ such that $d \circ b^* = b$, $e \circ c^* = c$, $b^*, c^* \in \mathcal{M}_1$ and (4) is a pushout in $\mathbf{C}$.

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "f"] \\ C \arrow[r, "c"] & D \end{tikzcd} \quad \begin{tikzcd} (A, a') \arrow[d, "f"'] \arrow[dr, phantom, "(2)"] & (E, e') \arrow[l, "d"'] \arrow[d, "g"] \\ (D, d') & (F, f') \arrow[l, "e"] \end{tikzcd} \quad \begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "(3)"] & E \arrow[l, "d"'] \arrow[d, "g"] \\ D & F \arrow[l, "e"] \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "b^*"] \arrow[d, "a"'] \arrow[dr, phantom, "(4)"] & E \arrow[d, "g"] \\ C \arrow[r, "c^*"] & F \end{tikzcd} \quad \begin{tikzcd} (B, b') \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(5)"] & (A, a') \arrow[d, "f"] \\ (C, c') \arrow[r, "c"] & (D, d') \end{tikzcd} \quad \begin{tikzcd} (B, b') \arrow[r, "b^*"] \arrow[d, "a"'] \arrow[dr, phantom, "(6)"] & (E, e') \arrow[d, "g"] \\ (C, c') \arrow[r, "c^*"] & (F, f') \end{tikzcd}$$

(i) If the diagram (5), corresponding to (1) in $\mathbf{C}$, is a valid extension of (1) in $X\backslash\mathbf{C}$, then the remark of Fact 2 implies that it is already a pushout in $X\backslash\mathbf{C}$. It remains to show that (6) is a valid square in $X\backslash\mathbf{C}$. With $d \circ b^* \circ b' = b \circ b' = a' = d \circ e'$ and d being a monomorphism it follows that $b^* \circ b = e'$ and thus $b^* \in X\backslash\mathbf{C}$, and analogously $c^* \in X\backslash\mathbf{C}$. This means the square (6), corresponding to (4), is also a pushout in $X\backslash\mathbf{C}$. Therefore (5) is the initial pushout over f in $X\backslash\mathbf{C}$.

(ii) If $a' : X \to A \in \mathcal{M}_1$ and the pushout complement of a' and f in $\mathbf{C}$ exists, we can construct the unique pushout complement (7) in $\mathbf{C}$, and with the remark of Fact 2 the corresponding diagram (8) is a pushout in $X\backslash\mathbf{C}$.

$$\begin{tikzcd} X \arrow[r, "a'"] \arrow[d, "h'"'] \arrow[dr, phantom, "(7)"] & A \arrow[d, "f"] \\ H \arrow[r, "h"] & D \end{tikzcd} \quad \begin{tikzcd} (X, id_X) \arrow[r, "a'"] \arrow[d, "h'"'] \arrow[dr, phantom, "(8)"] & (A, a') \arrow[d, "f"] \\ (H, h') \arrow[r, "h"] & (D, d') \end{tikzcd} \quad \begin{tikzcd} B \arrow[r, "b^*_X"] \arrow[d, "a"'] \arrow[dr, phantom, "(9)"] & X \arrow[d, "h'"] \\ C \arrow[r, "c^*_X"] & H \end{tikzcd}$$

It remains to show the initiality of (8). For any pushout (2), $e' : X \to E$ is unique with respect to $d \circ e' = a'$, because d is a monomorphism.
Since (1) is an initial pushout in $\mathbf{C}$ and (7) is a pushout, there are morphisms $b^*_X : B \to X$ and $c^*_X : C \to H$ such that $b^*_X, c^*_X \in \mathcal{M}_1$, $a' \circ b^*_X = b$, $h \circ c^*_X = c$ and (9) is a pushout. With $e \circ c^* \circ a = c \circ a = h \circ c^*_X \circ a = h \circ h' \circ b^*_X = f \circ a' \circ b^*_X = f \circ d \circ e' \circ b^*_X = e \circ g \circ e' \circ b^*_X$ and e being a monomorphism (9) implies that there is a unique $i : H \to F$ with $c^* = i \circ c^*_X$ and $i \circ h' = g \circ e'$. It

further follows that $e \circ i = h$ using the pushout properties of H. By pushout decomposition, then (10) is a pushout in $\mathbf{C}$ and using the remark of Fact 2 the corresponding square in $X\backslash\mathbf{C}$ is also a pushout. Therefore, (8) is an initial pushout over f in $X\backslash\mathbf{C}$.

$$
\begin{tikzcd}
B \arrow[r, "b^*_X"] \arrow[d, "a"] \arrow[rr, bend left, "b"] \arrow[dr, phantom, "(9)"] & X \arrow[r, "a'"] \arrow[d, "h'"] \arrow[rr, bend left, "e'"] \arrow[dr, phantom, "(7)"] & A \arrow[d, "f"] \arrow[dr, phantom, "(3)"] & E \arrow[l, "d"'] \arrow[d, "g"] \\
C \arrow[r, "c^*_X"] \arrow[rr, bend right, "c"'] & H \arrow[r, "h"] \arrow[rr, bend right, "i"'] & D & F \arrow[l, "e"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "e'"] \arrow[d, "h'"] \arrow[dr, phantom, "(10)"] & E \arrow[d, "g"] \\
H \arrow[r, "i"] & F
\end{tikzcd}
$$

4. Given $f : A \to D \in ComCat(F, G; \mathcal{I})$ with $f = (f_1, f_2) \in \mathcal{M}'_1 \times \mathcal{M}'_2$. Then we have initial pushouts $(1)_\mathbf{C}$ over $f_1 \in \mathcal{M}'_1$ in $\mathbf{C}$ with $b_1, c_1 \in \mathcal{M}_1$ and $(1)_\mathbf{D}$ over $f_2 \in \mathcal{M}'_2$ in $\mathbf{D}$ with $b_2, c_2 \in \mathcal{M}_2$.

$$
\begin{tikzcd}
B_1 \arrow[r, "b_1"] \arrow[d, "a_1"] \arrow[dr, phantom, "(1)_\mathbf{C}"] & A_1 \arrow[d, "f_1"] \\
C_1 \arrow[r, "c_1"] & D_1
\end{tikzcd}
\quad
\begin{tikzcd}
B_2 \arrow[r, "b_2"] \arrow[d, "a_2"] \arrow[dr, phantom, "(1)_\mathbf{D}"] & A_2 \arrow[d, "f_2"] \\
C_2 \arrow[r, "c_2"] & D_2
\end{tikzcd}
\quad
\begin{tikzcd}
(B_1, B_2, op_i^B) \arrow[r, "(b_1, b_2)"] \arrow[d, "(a_1, a_2)"] \arrow[dr, phantom, "(1)"] & (A_1, A_2, op_i^A) \arrow[d, "(f_1, f_2)"] \\
(C_1, C_2, op_i^C) \arrow[r, "(c_1, c_2)"] & (D_1, D_2, op_i^D)
\end{tikzcd}
$$

Since $G(\mathcal{M}_2) \subseteq Isos$, $G(b_2)^{-1}$ and $G(c_2)^{-1}$ exist. Define objects $B = (B_1, B_2, op_i^B = G(b_2)^{-1} \circ op_i^A \circ F(b_1))$ and $C = (C_1, C_2, op_i^C = G(c_1)^{-1} \circ op_i^D \circ F(c_1))$ in $ComCat(F, G; \mathcal{I})$. Then we have

- $G(b_2) \circ op_i^B = G(b_2) \circ G(b_2)^{-1} \circ op_i^A \circ F(b_1) = op_i^A \circ F(b_1)$,
- $G(c_2) \circ op_i^C = G(c_2) \circ G(c_2)^{-1} \circ op_i^D \circ F(c_1) = op_i^D \circ F(c_1)$,
- $G(c_2) \circ G(a_2) \circ op_i^B = G(f_2) \circ G(b_2) \circ op_i^B = G(f_2) \circ op_i^A \circ F(b_1) = op_i^D \circ F(f_1) \circ F(b_1) = op_i^D \circ F(c_1) \circ F(a_1) = G(c_2) \circ op_i^C \circ F(a_1)$ and $G(c_2)$ being an isomorphism implies that $G(a_2) \circ op_i^B = op_i^C \circ F(a_1)$,

which means that $a = (a_1, a_2)$, $b = (b_1, b_2)$ and $c = (c_1, c_2)$ are morphisms in $ComCat(F, G; \mathcal{I})$ with $b, c \in \mathcal{M}_1 \times \mathcal{M}_2$.
We shall show that (1) is an initial pushout over (f_1, f_2) in $ComCat(F, G; \mathcal{I})$. Fact 2 implies that (1) is a pushout with $(b_1, b_2), (c_1, c_2) \in \mathcal{M}_1 \times \mathcal{M}_2$.
It remains to show the initiality. For any pushout (2) in $ComCat(F, G; \mathcal{I})$ with $(d_1, d_2), (e_1, e_2) \in \mathcal{M}_1 \times \mathcal{M}_2$, Fact 2 implies that the components $(2)_\mathbf{C}$ and $(2)_\mathbf{D}$ are pushouts in $\mathbf{C}$ and $\mathbf{D}$, respectively.

$$
\begin{tikzcd}
(A_1, A_2, op_i^A) \arrow[d, "(f_1, f_2)"] \arrow[dr, phantom, "(2)"] & (E_1, E_2, op_i^E) \arrow[l, "(d_1, d_2)"'] \arrow[d, "(g_1, g_2)"] \\
(D_1, D_2, op_i^D) & (F_1, F_2, op_i^F) \arrow[l, "(e_1, e_2)"]
\end{tikzcd}
\quad
\begin{tikzcd}
A_1 \arrow[d, "f_1"] \arrow[dr, phantom, "(2)_\mathbf{C}"] & E_1 \arrow[l, "d_1"'] \arrow[d, "g_1"] \\
D_1 & F_1 \arrow[l, "e_1"]
\end{tikzcd}
\quad
\begin{tikzcd}
A_2 \arrow[d, "f_2"] \arrow[dr, phantom, "(2)_\mathbf{D}"] & E_2 \arrow[l, "d_2"'] \arrow[d, "g_2"] \\
D_2 & F_2 \arrow[l, "e_2"]
\end{tikzcd}
$$

The initiality of pushout $(1)_\mathbf{C}$ implies that there are unique morphisms $b_1^* : B_1 \to E_1$ and $c_1^* : C_1 \to F_1$ with $d_1 \circ b_1^* = b_1$, $e_1 \circ c_1^* = c_1$ and $b_1^*, c_1^* \in \mathcal{M}_1$ such that $(3)_\mathbf{C}$ is a pushout. Analogously, the initiality of pushout $(1)_\mathbf{D}$ implies that there are unique morphisms $b_2^* : B_2 \to E_2$ and $c_2^* : C_2 \to F_2$ with $d_2 \circ b_2^* = b_2$, $e_2 \circ c_2^* = c_2$ and $b_2^*, c_2^* \in \mathcal{M}_2$ such that $(3)_{\mathbf{D}}$ is a pushout.

$$\begin{tikzcd}
(B_1, B_2, op_i^B) \arrow[r, "(b_1^*, b_2^*)"] \arrow[d, "(a_1, a_2)"'] \arrow[dr, phantom, "(3)"] & (E_1, E_2, op_i^E) \arrow[d, "(g_1, g_2)"] \\
(C_1, C_2, op_i^C) \arrow[r, "(c_1^*, c_2^*)"'] & (F_1, F_2, op_i^F)
\end{tikzcd}
\quad
\begin{tikzcd}
B_1 \arrow[r, "b_1^*"] \arrow[d, "a_1"'] \arrow[dr, phantom, "(3)_{\mathbf{C}}"] & E_1 \arrow[d, "g_1"] \\
C_1 \arrow[r, "c_1^*"'] & F_1
\end{tikzcd}
\quad
\begin{tikzcd}
B_2 \arrow[r, "b_2^*"] \arrow[d, "a_2"'] \arrow[dr, phantom, "(3)_{\mathbf{D}}"] & E_2 \arrow[d, "g_2"] \\
C_2 \arrow[r, "c_2^*"'] & F_2
\end{tikzcd}$$

With $G(d_2) \circ G(b_2^*) \circ op_i^B = G(b_2) \circ op_i^B = op_i^A \circ F(b_1) = op_i^A \circ F(d_1) \circ F(b_1^*) = G(d_2) \circ op_i^E \circ F(b_1^*)$ and $G(d_2)$ being an isomorphism it follows that $(b_1^*, b_2^*) \in ComCat(F, G; \mathcal{I})$, and analogously $(c_1^*, c_2^*) \in ComCat(F, G; \mathcal{I})$. This means that we have unique morphisms $(b_1^*, b_2^*), (c_1^*, c_2^*) \in \mathcal{M}_1' \times \mathcal{M}_2' \cap ComCat(F, G; \mathcal{I})$ with $(d_1, d_2) \circ (b_1^*, b_2^*) = (b_1, b_2)$ and $(e_1, e_2) \circ (c_1^*, c_2^*) = (c_1, c_2)$, and by Fact 2 (3) composed of $(3)_{\mathbf{C}}$ and $(3)_{\mathbf{D}}$ is a pushout. Therefore (1) is the initial pushout over f in $ComCat(F, G; \mathcal{I})$.

5. Let $\mathcal{M}_1^{funct}$ denote the class of all $\mathcal{M}_1$-natural transformations. Given an $\mathcal{M}_1'$-natural transformation $f : A \to D$ in $[\mathbf{X}, \mathbf{C}]$, by assumption we can construct componentwise the initial pushout (1_x) over $f(x)$ in $\mathbf{C}$ for all $x \in \mathbf{X}$, with $b_0(x), c_0(x) \in \mathcal{M}_1$.

$$\begin{tikzcd}
B_0(x) \arrow[r, "b_0(x)"] \arrow[d, "a_0(x)"'] \arrow[dr, phantom, "(1_x)"] & A(x) \arrow[d, "f(x)"] \\
C_0(x) \arrow[r, "c_0(x)"'] & D(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
C_0(x) \arrow[rr, "c_0(x)"] \arrow[dr, "d_i'(x)"'] & (2) & D(x) \\
& C_i(x) \arrow[ur, "c_i(x)"'] &
\end{tikzcd}$$

Define $(C, (c_i' : C \to C_i)_{i \in I}, c : C \to D)$ as the limit in $[\mathbf{X}, \mathbf{C}]$ of all those $c_i : C_i \to D \in \mathcal{M}_1^{funct}$ such that for all $x \in \mathbf{X}$ there exists a $d_i'(x) : C_0(x) \to C_i(X) \in \mathcal{M}_1$ with $c_i(x) \circ d_i'(x) = c_0(x)$ (2), which defines the index set I. Limits in $[\mathbf{X}, \mathbf{C}]$ are constructed componentwise in $\mathbf{C}$, and if $\mathbf{C}$ has intersections of $\mathcal{M}_1$-subobjects it follows that also $[\mathbf{X}, \mathbf{C}]$ has intersections of $\mathcal{M}_1^{funct}$-subobjects. Hence $c_i' \in \mathcal{M}_1^{funct}$ and $c \in \mathcal{M}_1^{funct}$, and $C(x)$ is the limit of $c_i(x)$ in $\mathbf{C}$.
Now we construct the pullback (3) over $c \in \mathcal{M}_1^{funct}$ and f in $[\mathbf{X}, \mathbf{C}]$ and since $\mathcal{M}_1^{funct}$-morphisms are closed under pullbacks, also $b \in \mathcal{M}_1^{funct}$.

$$\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "a"'] \arrow[dr, phantom, "(3)"] & A \arrow[d, "f"] \\
C \arrow[r, "c"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0(x) \arrow[r, "b'(x)"] \arrow[rr, bend left, "b_0(x)"] \arrow[d, "a_0(x)"'] \arrow[dr, phantom, "(4_x)"] & B(x) \arrow[r, "b(x)"] \arrow[d, "a(x)"] \arrow[dr, phantom, "(3_x)"] & A(x) \arrow[d, "f(x)"] \\
C_0(x) \arrow[r, "c'(x)"] \arrow[dr, "d_i'(x)"'] \arrow[rr, bend right=60, "c_0(x)"'] & C(x) \arrow[r, "c(x)"] \arrow[d, "c_i'(x)"] & D(x) \\
& C_i(x) \arrow[ur, "c_i(x)"'] &
\end{tikzcd}$$

For $x \in X$, $C(x)$ being the limit of $c_i(x)$, the family $(d_i'(x))_{i \in I}$ with (2) implies that there is a unique morphism $c'(x) : C_0(x) \to C(x)$ with $c_i'(x) \circ c'(x) = d_i'(x)$ and

$c(x) \circ c'(x) = c_0(x)$. Then (3_x) being a pullback and $c(x) \circ c'(x) \circ a_0(x) = c_0(x) \circ a_0(x) = f(x) \circ b_0(x)$ implies the existence of a unique $b'(x) : B_0(x) \to B(x)$ with $b(x) \circ b'(x) = b_0(x)$ and $a(x) \circ b'(x) = c'(x) \circ a_0(x)$. $\mathcal{M}_1$ is closed under decomposition, $b_0(x) \in \mathcal{M}_1$ and $b(x) \in \mathcal{M}_1$ implies $b'(x) \in \mathcal{M}_1$. Since (1_x) is a pushout, (3_x) is a pullback, the whole diagram commutes and $c(x), b'(x) \in \mathcal{M}_1$, the $\mathcal{M}_1$ pushout-pullback property (see [3]) implies that (3_x) and (4_x) are both pushouts and pullbacks in $\mathbf{C}$ and hence (3) and (4) are both pushouts and pullbacks in $[\mathbf{X}, \mathbf{C}]$.

It remains to show the initiality of (3) over f. Given a pushout (5) with $b_1, c_1 \in \mathcal{M}_1^{funct}$ in $[\mathbf{X}, \mathbf{C}]$, (5_x) is a pushout in $\mathbf{C}$ for all $x \in \mathbf{X}$. Since (1_x) is an initial pushout in $\mathbf{C}$, there exist morphisms $b_1^*(x) : B_0(x) \to B_1(x)$, $c_1^* : C_0(x) \to C_1(x)$ with $b_1^*(x), c_1^*(x) \in \mathcal{M}_1$, $b_1(x) \circ b_1^*(x) = b_0(x)$ and $c_1(x) \circ c_1^*(x) = c_0(x)$. Hence $c_1(x)$ satisfies (2) for $i = 1$ and $d_1'(x) = c_1^*(x)$.

$$
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[dr, phantom, "(5)"] & B_1 \arrow[l, "b_1"'] \arrow[d, "a_1"] \\
D & C_1 \arrow[l, "c_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0(x) \arrow[r, "b_0(x)"] \arrow[d, "a_0(x)"'] \arrow[rr, bend left, "b_1^*(x)"] \arrow[dr, phantom, "(1_x)"] & A(x) \arrow[d, "f(x)"] \arrow[dr, phantom, "(5_x)"] & B_1(x) \arrow[l, "b_1(x)"'] \arrow[d, "a_1(x)"] \\
C_0(x) \arrow[r, "c_0(x)"'] \arrow[rr, bend right, "c_1^*(x)"'] & D(x) & C_1(x) \arrow[l, "c_1(x)"]
\end{tikzcd}
$$

This means c_1 is one of the morphisms the limit C was built of and there is a morphism $c_1' : C \to C_1$ with $c_1 \circ c_1' = c$ by construction of the limit C.

Since (5) is a pushout along $\mathcal{M}_1^{funct}$-morphisms it is also a pullback, and $f \circ b = c \circ a = c_1 \circ c_1' \circ a$ implies that there exists a unique $b_1' : B \to B_1$ with $b_1 \circ b_1' = b$ and $a_1 \circ b_1' = c_1' \circ a$. By $\mathcal{M}_1^{funct}$-decomposition also $b_1' \in \mathcal{M}_1^{funct}$. Now using also $c_1 \in \mathcal{M}_1^{funct}$ the $\mathcal{M}_1^{funct}$ pushout-pullback decomposition property implies that also (6) is a pushout, which shows the initiality of (3).

$$
\begin{tikzcd}
B \arrow[r, "b"] \arrow[d, "a"'] \arrow[rr, bend left, "b_1'"] \arrow[dr, phantom, "(3)"] & A \arrow[d, "f"] \arrow[dr, phantom, "(5)"] & B_1 \arrow[l, "b_1"'] \arrow[d, "a_1"] \\
C \arrow[r, "c"'] \arrow[rr, bend right, "c_1'"'] & D & C_1 \arrow[l, "c_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "b_1'"] \arrow[d, "a"'] \arrow[rr, bend left, "b"] \arrow[dr, phantom, "(6)"] & B_1 \arrow[r, "b_1"] \arrow[d, "a_1"] \arrow[dr, phantom, "(5)"] & A \arrow[d, "f"] \\
C \arrow[r, "c_1'"'] \arrow[rr, bend right, "c"'] & C_1 \arrow[r, "c_1"'] & D
\end{tikzcd}
$$
□

Example 5 According to the five cases in Theorem 2 we have the following examples based on Example 3, where in all cases except 3 $\mathcal{M}_1'$ is the class of all morphisms in $\mathbf{C}$.

1. Initial pushouts in **Graphs** × **Graphs** can be constructed componentwise in **Graphs** (see item 5).
2. $\mathbf{Graphs_{TG}}$ has initial pushouts to be constructed as in **Graphs**.
3. In **PSets** $\cong$ $\{1\}\backslash$**Sets**, we have three different cases for initial pushouts over a morphism $f : (A, a') \to (D, d')$:

   – Case 1: The morphism f is injective. Then (1) is the initial pushout over f in **Sets** with *inc* being an inclusion, which cannot be extended to a valid square in $\{1\}\backslash$**Sets**. But $a' : \{1\} \to A$ is injective and thus $a' \in \mathcal{M}_1$, and the pushout complement (2) of a' and f in **Sets** exists with $C = D\backslash f(A) \cup$

$f(a'(1)) \subseteq D$ and $d'(1) = f(a'(1))$, which means by construction in the proof of Theorem 2.3 that (3) is the initial pushout over f in $\{1\}\backslash\mathbf{Sets}$.

$$\begin{tikzcd} \varnothing \arrow[r, "\varnothing"] \arrow[d, "\varnothing"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "f"] \\ D\backslash f(A) \arrow[r, "inc"] & D \end{tikzcd} \quad \begin{tikzcd} \{1\} \arrow[r, "a'"] \arrow[d, "d'"'] \arrow[dr, phantom, "(2)"] & A \arrow[d, "f"] \\ C \arrow[r, "inc"] & D \end{tikzcd} \quad \begin{tikzcd} (\{1\}, id) \arrow[r, "a'"] \arrow[d, "d'"'] \arrow[dr, phantom, "(3)"] & (A, a') \arrow[d, "f"] \\ (C, d') \arrow[r, "inc"] & (D, d') \end{tikzcd}$$

- Case 2: f is noninjective, and $a'(1) \in Id(f)$, with $Id(f) = \{x \in A \mid \exists y \in A, x \neq y : f(x) = f(y)\}$. Then (4) with $C = D\backslash f(A) \cup f(Id(f))$ is the initial pushout over f in **Sets**, which due to $d'(1) = f(a'(1)) \in C$ can be extended to a valid diagram (5) and hence by construction to the initial pushout (5) over f in $\{1\}\backslash\mathbf{Sets}$.

$$\begin{tikzcd} Id(f) \arrow[r, "inc"] \arrow[d, "f|_{Id(f)}"'] \arrow[dr, phantom, "(4)"] & A \arrow[d, "f"] \\ C \arrow[r, "inc"] & D \end{tikzcd} \qquad \begin{tikzcd} (Id(f), a') \arrow[r, "inc"] \arrow[d, "f|_{Id(f)}"'] \arrow[dr, phantom, "(5)"] & (A, a') \arrow[d, "f"] \\ (C, d') \arrow[r, "inc"] & (D, d') \end{tikzcd}$$

- Case 3: f is noninjective, and $a'(1) \notin Id(f)$. In this case, pushout (4) above is the initial pushout over f in **Sets**, but it cannot be extended to a valid square in $\{1\}\backslash\mathbf{Sets}$. Moreover, the pushout complement over a' and f does not exist in **Sets**, thus Theorem 2 cannot be applied. Nevertheless, the initial pushout in this case in $\{1\}\backslash\mathbf{Sets}$ exists, it is the following pushout (6) with $C = D\backslash f(A) \cup f(Id(f)) \cup d'(1)$, which generalises cases 1 and 2.

$$\begin{tikzcd} (Id(f) \cup a'(1), a') \arrow[r, "inc"] \arrow[d, "f|_{Id(f) \cup a'(1)}"'] \arrow[dr, phantom, "(6)"] & (A, a') \arrow[d, "f"] \\ (C, d') \arrow[r, "inc"] & (D, d') \end{tikzcd}$$

4. The constructions in product and slice categories are examples for the construction of initial pushouts in comma categories; hence we can reuse the examples in items 1 and 2.
5. Since **Sets** has intersections of $\mathcal{M}$-subobjects for the class $\mathcal{M}$ of injective functions, the category **Graphs** as a functor category of **Sets** has initial pushouts.

3.2 Finite Coproducts Compatible with $\mathcal{M}$

In the double pushout graph transformation, for parallel productions we need not only finite coproducts for the definition of the objects of the parallel production, but also finite coproducts compatible with $\mathcal{M}$ which ensure that the morphisms of the parallel productions are $\mathcal{M}$-morphisms. For the existence of finite coproducts compatible with $\mathcal{M}$ it is sufficient to show the existence of binary coproducts compatible with $\mathcal{M}$.

Definition 4 (Binary Coproduct Compatible with $\mathcal{M}$). A (weak) adhesive HLR category $(\mathbf{C}, \mathcal{M})$ has *binary coproducts compatible with* $\mathcal{M}$ if $\mathbf{C}$ has binary coproducts and, for each pair of morphisms $f : A \to A'$, $g : B \to B'$ with $f, g \in \mathcal{M}$, the coproduct morphism is also an $\mathcal{M}$-morphism, i.e. $f + g : A + B \to A' + B' \in \mathcal{M}$.

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"] & A+B \arrow[d, "f+g"] & B \arrow[l, "i_B"'] \arrow[d, "g"] \\ A' \arrow[r, "i_{A'}"] & A'+B' & B' \arrow[l, "i_{B'}"'] \end{tikzcd}$$

Example 6 **Sets**, **Graphs**, $\mathbf{Graphs_{TG}}$ and $\mathbf{AGraphs_{ATG}}$ have binary coproducts compatible with $\mathcal{M}$. For **Graphs** and $\mathbf{Graphs_{TG}}$ this follows by Thm. 3, for $\mathbf{AGraphs_{ATG}}$ see [3].

Theorem 3 (Preservation of Bin. Coproducts Compatible with $\mathcal{M}$). *Given* (*weak*) *adhesive HLR categories* $(\mathbf{C}, \mathcal{M}_1)$ *and* $(\mathbf{D}, \mathcal{M}_2)$ *with binary coproducts compatible with* $\mathcal{M}_1$ *and* $\mathcal{M}_2$*, respectively. Then we have the following results.*

1. $\mathbf{C} \times \mathbf{D}$ *has binary coproducts compatible with* $\mathcal{M}_1 \times \mathcal{M}_2$.
2. $\mathbf{C}\backslash X$ *has binary coproducts compatible with* $\mathcal{M}_1 \cap \mathbf{C}\backslash X$.
3. $X\backslash\mathbf{C}$ *has binary coproducts compatible with* $X\backslash\mathbf{C} \cap \mathcal{M}_1$*, if* $\mathbf{C}$ *has general pushouts.*
4. *If F preserves coproducts, then* $ComCat(F, G; \mathcal{I})$ *has binary coproducts compatible with* $\mathcal{M}_1 \times \mathcal{M}_2$*-morphisms.*
5. $[\mathbf{X}, \mathbf{C}]$ *has binary coproducts compatible with* $\mathcal{M}_1$*-natural transformations* (*see remark after Theorem* 2).

Proof

1. Since $\mathbf{C} \times \mathbf{D} \tilde{=} ComCat(!_\mathbf{C} : \mathbf{C} \to \mathbf{1}, !_\mathbf{D} : \mathbf{D} \to \mathbf{1}, \varnothing)$ and $!_\mathbf{C}$ preserves coproducts this follows from item 4. The coproduct of objects (A_1, A_2) and (B_1, B_2) of the product category is the componentwise coproduct $(A_1 + B_1, A_2 + B_2)$ in $\mathbf{C}$ and $\mathbf{D}$, respectively. Analogously, the coproduct of morphism $f = (f_1, f_2)$ and $g = (g_1, g_2)$ is the componentwise coproduct morphism $f + g = (f_1 + g_1, f_2 + g_2)$ in $\mathbf{C}$ and $\mathbf{D}$. If $f, g \in \mathcal{M}_1 \times \mathcal{M}_2$, then also $f + g \in \mathcal{M}_1 \times \mathcal{M}_2$.
2. Since $\mathbf{C}\backslash X \tilde{=} ComCat(id_\mathbf{C} : \mathbf{C} \to \mathbf{C}, X : \mathbf{1} \to \mathbf{C}, \{1\})$ and $id_\mathbf{C}$ preserves coproducts this follows from item 4. In the slice category, the coproduct of (A, a') and (B, b') is the object $(A + B, [a', b'])$ which consists of the coproduct $A + B$ in $\mathbf{C}$ together with the morphism $[a', b'] : A + B \to X$ induced by a' and b'. Analogously, for morphisms f and g the coproduct morphism in $\mathbf{C}\backslash X$ is the coproduct morphism $f + g$ in $\mathbf{C}$. If $f, g \in \mathcal{M}_1 \cap \mathbf{C}\backslash X$, then also $f + g \in \mathcal{M}_1 \cap \mathbf{C}\backslash X$.
3. If $\mathbf{C}$ has general pushouts, given two objects (A, a') and (B, b') in $X\backslash\mathbf{C}$ we construct the pushout (1) over a' and b' in $\mathbf{C}$. The coproduct of (A, a') and (B, b') is the pushout object $A +_X B$. Given morphisms $f : A \to C$ and $g : B \to D$ in $X\backslash\mathbf{C}$, we can construct the pushouts (1), (2), (3) and (4) in $\mathbf{C}$, and by pushout composition we have $G \tilde{=} C +_X D$ leading to the coproduct morphism $f +_X g$

in $X\backslash\mathbf{C}$. Using the Butterfly Lemma (see [3, 10]), also (5) is a pushout in $\mathbf{C}$. If $f, g \in \mathcal{M}_1$, $f + g \in \mathcal{M}_1$ because $\mathbf{C}$ has binary coproducts compatible with $\mathcal{M}_1$. Since $\mathcal{M}_1$-morphisms are closed under pushouts, it follows that also $f +_X g \in \mathcal{M}_1$.

$$\begin{tikzcd}
X \arrow[r, "b'"] \arrow[d, "a'"] \arrow[rr, bend left, "d'"] \arrow[dd, bend right, "c'"'] \arrow[dr, phantom, "(1)"] & B \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "(2)"] & D \arrow[d] \\
A \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "(3)"] & A +_X B \arrow[r] \arrow[d] \arrow[dr, "f+_X g"] \arrow[dr, phantom, "(4)"'] & E \arrow[d] \\
C \arrow[r] & F \arrow[r] & G \cong C +_X D
\end{tikzcd}
\qquad
\begin{tikzcd}
A + B \arrow[r, "f+g"] \arrow[d] \arrow[dr, phantom, "(5)"] & C + D \arrow[d] \\
A +_X B \arrow[r, "f+_X g"] & C +_X D
\end{tikzcd}$$

4. If $\mathbf{C}$ and $\mathbf{D}$ have binary coproducts and F preserves coproducts, then the coproduct of two objects $A = (A_1, A_2, op_i^A)$ and $B = (B_1, B_2, op_i^B)$ in $ComCat(F, G; \mathcal{I})$ is the object $A + B = (A_1 + B_1, A_2 + B_2, op_i^{A+B})$, where op_i^{A+B} is the unique morphism induced by $G(i_{A_2}) \circ op_i^A$ and $G(i_{B_2}) \circ op_i^B$. If G preserves coproduct, i.e. $G(A_2)+G(B_2)=G(A_2+B_2)$, then $op_i^{A+B}=op_i^A+op_i^B$.

$$\begin{tikzcd}
F(A_1) \arrow[r, "F(i_{A_1})"] \arrow[d, "op_i^A"] & F(A_1 + B_1) \arrow[d, "op_i^{A+B}"] & F(B_1) \arrow[l, "F(i_{B_1})"'] \arrow[d, "op_i^B"] \\
G(A_2) \arrow[r, "G(i_{A_2})"] & G(A_2 + B_2) & G(B_2) \arrow[l, "G(i_{B_2})"']
\end{tikzcd}$$

For morphisms $f = (f_1, f_2) : (A_1, A_2, op_i^A) \to (C_1, C_2, op_i^C)$ and $g = (g_1, g_2) : (B_1, B_2, op_i^B) \to (D_1, D_2, op_i^D)$ we get a coproduct morphism $f + g = (f_1 + g_1, f_2 + g_2)$. If $f, g \in \mathcal{M}_1 \times \mathcal{M}_2$, the compatibility of coproducts with $\mathcal{M}_1$ and $\mathcal{M}_2$ in $\mathbf{C}$ and $\mathbf{D}$ ensures that $f_1 + g_1 \in \mathcal{M}_1$ and $f_2 + g_2 \in \mathcal{M}_2$, respectively, that means $f + g \in \mathcal{M}_1 \times \mathcal{M}_2$.

5. If $\mathbf{C}$ has binary coproduct, the coproduct of two functors $A, B : \mathbf{X} \to \mathbf{C}$ in $[\mathbf{X}, \mathbf{C}]$ is the componentwise coproduct functor $A + B$ with $A + B(x) = A(x) + B(x)$ for an object $x \in \mathbf{X}$ and $A + B(h) = A(h) + B(h)$ for a morphism $h \in \mathbf{X}$. For natural transformations $f = (f_x)_{x\in\mathbf{X}}$ and $g = (g_x)_{x\in\mathbf{X}}$, the coproduct morphism is the componentwise coproduct morphism in $\mathbf{C}$, i.e. $f + g = (f_x + g_x)_{x\in\mathbf{X}}$. If f and g are $\mathcal{M}_1$-natural transformations we have $f_x, g_x \in \mathcal{M}_1$ for all $x \in \mathbf{C}$, and since $\mathbf{C}$ has binary coproducts compatible with $\mathcal{M}_1$ it follows that also $f_x + b_x \in \mathcal{M}_1$ for all $x \in \mathbf{X}$, therefore $f + g$ is an $\mathcal{M}_1$-natural transformation. □

Example 7

1. In **Graphs** × **Graphs**, binary coproducts are constructed componentwise and are compatible with injective morphisms.
2. In $\mathbf{Graphs_{TG}}$, binary coproducts are constructed in **Graphs** and lifted to $\mathbf{Graphs_{TG}}$. From the coproducts in **Graphs** they inherit the compatibility with injective morphisms.

3. The construction of binary coproducts in **PSets** $\cong \{1\}\backslash$**Sets** is given by the corresponding pushout in **Sets**, i.e. the binary coproduct of two sets A and B with distinguished elements 1_A and 1_B, representing the respective undefined, is the set $A\backslash\{1_A\} \dot{\cup} B\backslash\{1_B\} \dot{\cup} \{1\}$, where 1 represents the new undefined element of the coproduct.
4. In **PTNets**, binary coproducts are constructed componentwise in **Sets** and are compatible with injective Petri net morphisms.
5. In **Graphs**, binary coproducts are constructed componentwise in **Sets** and are compatible with injective graph morphisms.

3.3 $\mathcal{E}'$-$\mathcal{M}'$ Pair Factorization

A possible approach to analyze the confluence of a transformation system is to show the termination of the system, and the strict confluence of so-called critical pairs. The concept of an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization is essential for the definition of critical pairs.

Definition 5 ($\mathcal{E}'$–$\mathcal{M}'$ Pair Factorization). Given a class of morphism pairs $\mathcal{E}'$ with the same codomain and a class $\mathcal{M}'$ of morphisms, a (weak) adhesive HLR category has an $\mathcal{E}'$–$\mathcal{M}'$ *pair factorization* if, for each pair of morphisms $f : A \to C$ and $g : B \to C$, there exist an object K and morphisms $e : A \to K$, $e' : B \to K$, and $m : K \to C$ with $(e, e') \in \mathcal{E}'$ and $m \in \mathcal{M}'$ such that $m \circ e = f$ and $m \circ e' = g$:

$$\begin{tikzcd}
A \arrow[dr, "e"] \arrow[drr, bend left, "f"] & & \\
 & K \arrow[r, "m"] & C \\
B \arrow[ur, "e'"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Definition 6 (Strong $\mathcal{E}'$–$\mathcal{M}'$ Pair Factorization). An $\mathcal{E}'$–$\mathcal{M}'$ pair factorization is called *strong*, if the following $\mathcal{E}'$–$\mathcal{M}'$ diagonal property holds:

Given $(e, e') \in \mathcal{E}'$, $m \in \mathcal{M}'$, and morphisms a, b, n as shown in the following diagram, with $n \circ e = m \circ a$ and $n \circ e' = m \circ b$, then there exists a unique $d : K \to L$ such that $m \circ d = n$, $d \circ e = a$ and $d \circ e' = b$.

$$\begin{tikzcd}
A \arrow[drr, "e"] \arrow[ddr, "a"'] & & \\
 & B \arrow[r, "e'"] \arrow[d, "b"] & K \arrow[d, "n"] \arrow[dl, dotted, "d"] \\
 & L \arrow[r, "m"] & C
\end{tikzcd}$$

Fact 4 (Strong $\mathcal{E}'$–$\mathcal{M}'$ Pair Factorization). *In a* (*weak*) *adhesive HLR category* (**C**, $\mathcal{M}$)*, the following properties hold*:

1. *If* (**C**, $\mathcal{M}$) *has a strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorization, then the $\mathcal{E}'$–$\mathcal{M}'$ pair factorization is unique up to isomorphism.*
2. *A strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorization is functorial, i.e. given morphisms $a, b, c, f_1, g_1, f_2, g_2$ as shown in the following diagram with $c \circ f_1 = f_2 \circ a$ and*

$c \circ g_1 = g_2 \circ b$, and $\mathcal{E}'$–$\mathcal{M}'$ pair factorizations $((e_1, e'_1), m_1)$ and $((e_2, e'_2), m_2)$ of f_1, g_1 and f_2, g_2, respectively, then there exists a unique $d : K_1 \to K_2$ such that $d \circ e_1 = e_2 \circ a$, $d \circ e'_1 = e'_2 \circ b$ and $c \circ m_1 = m_2 \circ d$.

$$
\begin{tikzcd}
& A_1 \arrow[dr, "e_1"] \arrow[drr, bend left, "f_1"] \arrow[ddd, "a"] & & \\
& & K_1 \arrow[r, "m_1"] \arrow[ddd, dotted, "d"] & C_1 \arrow[ddd, "c"] \\
B_1 \arrow[urr, "e'_1"] \arrow[urrr, bend right, "g_1"'] \arrow[ddd, "b"] & & & \\
& A_2 \arrow[dr, "e_2"] \arrow[drr, bend left, "f_2"] & & \\
& & K_2 \arrow[r, "m_2"] & C_2 \\
B_2 \arrow[urr, "e'_2"] \arrow[urrr, bend right, "g_2"'] & & &
\end{tikzcd}
$$

Proof

1. We show that strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorizations are unique up to isomorphism. Suppose $((e_1, e'_1), m_1)$ with $m_1 : K_1 \to C$ and $((e_2, e'_2), m_2)$ with $m_2 : K_2 \to C$ are two $\mathcal{E}'$–$\mathcal{M}'$ pair factorizations of f and g.

$$
\begin{tikzcd}
A \arrow[r, "e_1"] \arrow[drr, bend left, "f"] \arrow[ddr, "e_2"'] & K_1 \arrow[dr, "m_1"] \arrow[dd, shift left, "k"] & \\
& & C \\
B \arrow[r, "e'_2"'] \arrow[urr, bend right, "g"'] \arrow[uur, "e'_1"] & K_2 \arrow[ur, "m_2"'] \arrow[uu, shift left, "k'"] &
\end{tikzcd}
$$

Using the morphisms $(e_1, e'_1) \in \mathcal{E}'$ and $m_1 \in \mathcal{M}'$, from the $\mathcal{E}'$–$\mathcal{M}'$ diagonal property we obtain a unique morphism $k : K_1 \to K_2$ with $m_2 \circ k = m_1$, $k \circ e_1 = e_2$ and $k \circ e'_1 = e'_2$. At the same time, id_{K_1} is unique with respect to $m_1 \circ id_{K_1} = m_1$, $id_{K_1} \circ e_1 = e_1$ and $id_{K_1} \circ e'_1 = e'_1$.
By exchanging the roles of $((e_1, e'_1), m_1)$ and $((e_2, e'_2), m_2)$, the $\mathcal{E}'$–$\mathcal{M}'$ diagonal property implies that there is a unique $k' : K_2 \to K_1$ with $m_1 \circ k' = m_2$, $k' \circ e_2 = e_1$ and $k' \circ e'_2 = e'_1$. Also, id_{K_2} is unique with respect to $m_2 \circ id_{K_2} = m_2$, $id_{K_2} \circ e_2 = e_2$ and $id_{K_2} \circ e'_2 = e'_2$.
Thus, from $m_1 \circ k' \circ k = m_2 \circ k = m_1$, $k' \circ k \circ e_1 = k' \circ e_2 = e_1$ and $k' \circ k \circ e'_1 = k' \circ e'_2 = e'_1$ it follows that $k' \circ k = id_{K_1}$, and analogously $k \circ k' = id_{K_2}$. This means that K_1 and K_2 as well as the corresponding morphisms are isomorphic.
2. The fact that a strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorization is functorial follows directly from the $\mathcal{E}'$–$\mathcal{M}'$ diagonal property: Given the setting above, since $(e_1, e'_2) \in \mathcal{E}'$

and $m_2 \in \mathcal{M}'$ we obtain a unique morphism $d : K_1 \to K_2$ with $m_2 \circ d = c \circ m_1$, $d \circ e_1 = e_2 \circ a$ and $d \circ e'_1 = e'_2 \circ b$. □

Example 8

- In **Sets**, one possible choice for $\mathcal{E}'$ and $\mathcal{M}'$ is to define $\mathcal{E}'$ as the class of jointly surjective morphisms and $\mathcal{M}'$ as the class of injective morphisms leading to a strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorization.
- In categories with binary coproducts and an $\mathcal{E}$–$\mathcal{M}$ factorization we have an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization with $\mathcal{M}' = \mathcal{M}$ and $(e, e') \in \mathcal{E}' \Leftrightarrow [e, e'] \in \mathcal{E}$, where $[e, e']$ is the morphism induced by the binary coproduct.
- In the category $\mathbf{AGraphs_{ATG}}$ there are different possible choices for an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization (see [3]).

Theorem 4 (Preservation of $\mathcal{E}'$–$\mathcal{M}'$ Pair Factorizations). *Given (weak) adhesive HLR categories $(\mathbf{C}, \mathcal{M}_1)$ and $(\mathbf{D}, \mathcal{M}_2)$ with $\mathcal{E}'_1$–$\mathcal{M}'_1$ and $\mathcal{E}'_2$–$\mathcal{M}'_2$ pair factorizations, respectively. Then we have the following results.*

1. *$\mathbf{C} \times \mathbf{D}$ has an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization with $\mathcal{M}' = \mathcal{M}'_1 \times \mathcal{M}'_2$ and $\mathcal{E}' = \{((e_1, e_2), (e'_1, e'_2)) | (e_1, e'_1) \in \mathcal{E}'_1, (e_2, e'_2) \in \mathcal{E}'_2\}$. If the $\mathcal{E}'_1$–$\mathcal{M}'_1$ and $\mathcal{E}'_2$–$\mathcal{M}'_2$ pair factorizations are strong, so is the $\mathcal{E}'$–$\mathcal{M}'$ pair factorization.*
2. *$\mathbf{C}\backslash X$ has an $\mathcal{E}'_1$–$\mathcal{M}'_1$ pair factorization, where strongness is preserved.*
3. *If $\mathcal{M}'_1$ is a class of monomorphisms then $X\backslash\mathbf{C}$ has an $\mathcal{E}'_1$–$\mathcal{M}'_1$ pair factorization, where strongness is preserved.*
4. *If $G(\mathcal{M}'_2) \subseteq Isos$, then $ComCat(F, G; \mathcal{I})$ has an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization (with $\mathcal{E}'$, $\mathcal{M}'$ as in the product category), where strongness is preserved.*
5. *If $\mathcal{E}'_1$–$\mathcal{M}'_1$ is a strong pair factorization in $\mathbf{C}$, then $[\mathbf{X}, \mathbf{C}]$ has a strong $\mathcal{E}'^{funct}_1$–$\mathcal{M}'^{funct}_1$ pair factorization, where $\mathcal{M}'^{funct}_1$ is the class of $\mathcal{M}'_1$-natural transformations and $(e, e') \in \mathcal{E}'^{funct}_1$ iff $(e(x), e'(x)) \in \mathcal{E}'_1$ for all $x \in \mathbf{X}$.*

Proof

1. Since $\mathbf{C} \times \mathbf{D} \stackrel{\sim}{=} ComCat(!_\mathbf{C} : \mathbf{C} \to \mathbf{1}, !_\mathbf{D} : \mathbf{D} \to \mathbf{1}, \varnothing)$ and $!_\mathbf{D}(\mathcal{M}'_2) \subseteq \{id_1\} = Isos$ this follows from item 4. For morphisms $f = (f_1, f_2)$ and $g = (g_1, g_2)$ in $\mathbf{C} \times \mathbf{D}$ we construct the componentwise pair factorizations $((e_1, e'_1), m_1)$ of f_1, g_1 with $(e_1, e'_1) \in \mathcal{E}'_1$ and $m_1 \in \mathcal{M}'_1$ and $((e_2, e'_2), m_2)$ of f_2, g_2 with $(e_2, e'_2) \in \mathcal{E}'_2$ and $m_2 \in \mathcal{M}'_2$. This leads to morphisms $e = (e_1, e_2), e' = (e'_1, e'_2)$ and $m = (m_1, m_2)$ in $\mathbf{C} \times \mathbf{D}$ and an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization with $(e, e') \in \mathcal{E}'$ and $m \in \mathcal{M}$. If the $\mathcal{E}'_1$–$\mathcal{M}'_1$ and the $\mathcal{E}'_2$–$\mathcal{M}'_2$ pair factorizations are strong, then also $\mathcal{E}'$–$\mathcal{M}'$ is a strong pair factorization.
2. Since $\mathbf{C}\backslash X \stackrel{\sim}{=} ComCat(id_\mathbf{C} : \mathbf{C} \to \mathbf{C}, X : \mathbf{1} \to \mathbf{C}, \{1\})$ and $X(\mathcal{M}'_2) \subseteq X(\{id_1\}) = \{id_X\} \subseteq Isos$ this follows from item 4. Given morphisms f and g in $\mathbf{C}\backslash X$, an $\mathcal{E}'_1$–$\mathcal{M}'_1$ pair factorization of f and g in $\mathbf{C}$ is also an $\mathcal{E}'_1$–$\mathcal{M}'_1$ of f and g in $\mathbf{C}\backslash X$. If the $\mathcal{E}'_1$–$\mathcal{M}'_1$ pair factorization is strong in $\mathbf{C}$, this is also true for $\mathbf{C}\backslash X$.
3. Given morphisms $f : (A, a') \to (C, c')$ and $g : (B, b') \to (C, c')$ in $X\backslash\mathbf{C}$, we have an $\mathcal{E}'_1 - \mathcal{M}'_1$ pair factorization $((e, e'), m)$ of f and g in $\mathbf{C}$. This is also a pair factorization in $X\backslash\mathbf{C}$ if $e \circ a' = e' \circ b'$, because then $(K, e \circ a')$ and $(K, e' \circ b')$ is

the same object in $X\backslash\mathbf{C}$. If m is a monomorphism, this follows from $m \circ e \circ a' = f \circ a' = c' = g \circ b' = m \circ e' \circ b'$.

$$\begin{tikzcd}
& A \arrow[dr, "e"] \arrow[drr, bend left, "f"] & & \\
X \arrow[ur, "a'"] \arrow[dr, "b'"'] & & K \arrow[r, "m"] & C \\
& B \arrow[ur, "e'"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

To show that strongness is preserved, we have to show the $\mathcal{E}'_1$–$\mathcal{M}'_1$ diagonal property in $X\backslash\mathbf{C}$. Since it holds in $\mathbf{C}$, given $(e, e') \in \mathcal{E}'_1 \cap X\backslash\mathbf{C}$, $m \in \mathcal{M}'_1 \cap X\backslash\mathbf{C}$ and morphisms a, b, n in $X\backslash\mathbf{C}$ with $n \circ e = m \circ a$ and $n \circ e' = m \circ b$ we get an induced unique $d : K \to L$ with $d \circ e = a$, $d \circ e' = b$ and $m \circ d = n$. It remains to show that d is a valid morphism in $X\backslash\mathbf{C}$.

$$\begin{tikzcd}
(A, a') \arrow[drr, "e"] \arrow[ddr, "a"'] & & \\
& (B, b') \arrow[r, "e'"] \arrow[d, "b"] & (K, k') \arrow[dl, dotted, "d"'] \arrow[d, "n"] \\
& (L, l') \arrow[r, "m"] & (C, c')
\end{tikzcd}$$

Since $m \circ d \circ k' = n \circ k' = c' = m \circ l'$ and m is a monomorphisms it follows that $d \circ k' = l'$ and thus $d \in X\backslash\mathbf{C}$.

4. Given objects $A = (A_1, A_2, op_i^A)$, $B = (B_1, B_2, op_i^B)$, $C = (C_1, C_2, op_i^C)$ and morphisms $f = (f_1, f_2) : A \to C$, $g = (g_1, g_2) : B \to C$ in $ComCat(F, G; \mathcal{I})$, we have an $\mathcal{E}'_1$-$\mathcal{M}'_1$ pair factorization $((e_1, e'_1), m_1)$ of f_1, g_1 with $m_1 : K_1 \to C_1$ in $\mathbf{C}$ and an $\mathcal{E}'_2$-$\mathcal{M}'_2$ pair factorization $((e_2, e'_2), m_2)$ of f_2, g_2 with $m_2 : K_2 \to C_2$ in $\mathbf{D}$. If $G(m_2)$ is an isomorphism, we have an object $K = (K_1, K_2, op_i^K = G(m_2)^{-1} \circ op_i^C \circ F(m_1))$ in $ComCat(F, G; \mathcal{I})$. By definition, $m = (m_1, m_2) : K \to C$ is a morphism in $ComCat(F, G; \mathcal{I})$. For $e = (e_1, e_2)$ we have $op_i^K \circ F(e_1) = G(m_2)^{-1} \circ op_i^C x \circ F(m_1) \circ F(e_1) = G(m_2)^{-1} \circ op_i^C \circ F(f_1) = G(m_2)^{-1} \circ G(f_2) \circ op_i^A = G(e_2) \circ op_i^A$ and an analogous result for $e' = (e'_1, e'_2)$, therefore e and e' are morphisms in $ComCat(F, G; \mathcal{I})$. This means, $((e, e'), m)$ is an $\mathcal{E}'$-$\mathcal{M}'$ pair factorization in $ComCat(F, G; \mathcal{I})$.

$$\begin{tikzcd}
& F(A_1) \arrow[dr, "F(e_1)"] \arrow[drr, bend left, "F(f_1)"] \arrow[ddd, "op_i^A"] & & \\
& & F(K_1) \arrow[r, "F(m_1)"] \arrow[ddd, "op_i^K = G(m_2)^{-1} \circ op_i^C \circ F(m_1)"] & F(C_1) \arrow[ddd, "op_i^C"] \\
F(B_1) \arrow[urr, "F(e'_1)"] \arrow[urrr, bend right, "F(g_1)"'] \arrow[ddd, "op_i^B"] & & & \\
& G(A_2) \arrow[dr, "G(e_2)"] \arrow[drr, bend left, "G(f_2)"] & & \\
& & G(K_2) \arrow[r, "G(m_2)"] & G(C_2) \\
G(B_2) \arrow[urr, "G(e'_2)"] \arrow[urrr, bend right, "G(g_2)"'] & & &
\end{tikzcd}$$

To show the $\mathcal{E}'$–$\mathcal{M}'$ diagonal property, we consider $(e, e') = ((e_1, e_2), (e'_1, e'_2)) \in \mathcal{E}'$, $m = (m_1, m_2) \in \mathcal{M}'$ and morphisms $a = (a_1, a_2)$, $b = (b_1, b_2)$, $n = (n_1, n_2)$ in $ComCat(F, G; \mathcal{I})$.

$$
\begin{tikzcd}
(A_1, A_2, op_i^A) \arrow[drrr, "{(e_1,e_2)}"] \arrow[ddr, "{(a_1,a_2)}"'] & & & \\
& (B_1, B_2, op_i^B) \arrow[rr, "{(e'_1,e'_2)}"] \arrow[d, "{(b_1,b_2)}"] & & (K_1, K_2, op_i^K) \arrow[d, "{(n_1,n_2)}"] \arrow[dll, dashed, "{(d_1,d_2)}"'] \\
& (L_1, L_2, op_i^L) \arrow[rr, "{(m_1,m_2)}"] & & (C_1, C_2, op_i^C)
\end{tikzcd}
$$

Since $(e_1, e'_1) \in \mathcal{E}'_1$ and $m_1 \in \mathcal{M}'_1$, we get a uniqe morphism $d_1 : K_1 \to L_1$ in $\mathbf{C}$ with $m_1 \circ d_1 = n_1$, $d_1 \circ e_1 = a_1$ and $d_1 \circ e'_1 = b_1$. Analogously, the $\mathcal{E}'_2$–$\mathcal{M}'_2$ diagonal property implies a unique $d_2 : K_2 \to L_2$ with $m_2 \circ d_2 = n_2$, $d_2 \circ e_2 = a_2$ and $d_2 \circ e'_2 = b_2$.
It remains to show that $d = (d_1, d_2) \in ComCat(F, G; \mathcal{I})$, i.e. the compatibility with the operations. We have for all $i \in I$ $G(m_2) \circ op_i^L \circ F(d_1) = op_i^C \circ F(m_1) \circ F(d_1) = op_i^C \circ F(n_1) = G(n_2) \circ op_i^K = G(m_2) \circ G(d_2) \circ op_i^K$, and since $G(m_2)$ is an isomorphism it follows that $op_i^L \circ F(d_1) = G(d_2) \circ op_i^K$, i.e. $d \in ComCat(F, G; \mathcal{I})$.

$$
\begin{tikzcd}
& F(K_1) \arrow[rr, "op_i^K"] \arrow[dl, "F(d_1)"'] \arrow[dd, "F(n_1)"] & & G(K_2) \arrow[dr, "G(d_2)"] \arrow[dd, "G(n_2)"'] & \\
F(L_1) \arrow[rrrr, bend left, "op_i^L"] \arrow[dr, "F(m_1)"'] & & & & G(L_2) \arrow[dl, "G(m_2)"] \\
& F(C_1) \arrow[rr, "op_i^C"] & & G(C_2) &
\end{tikzcd}
$$

5. Given morphisms $f = (f(x))_{x\in\mathbf{X}}$ and $g = (g(x))_{x\in\mathbf{X}}$ in $[\mathbf{X}, \mathbf{C}]$, we have an $\mathcal{E}'_1$–$\mathcal{M}'_1$ pair factorization $((e_x, e'_x), m_x)$ with $m_x : K_x \to C(x)$ of $f(x), g(x)$ in $\mathbf{C}$ for all $x \in \mathbf{X}$. We have to show that $K(x) = K_x$ can be extended to a functor and that $e = (e_x)_{x\in\mathbf{X}}$, $e' = (e'_x)_{x\in\mathbf{X}}$ and $m = (m_x)_{x\in\mathbf{X}}$ are natural transformations.
For a morphism $h : x \to y$ in $\mathbf{X}$, we use the $\mathcal{E}'_1$–$\mathcal{M}'_1$ diagonal property in $\mathbf{C}$ and $(e_x, e'_x) \in \mathcal{E}'_1$, $m_y \in \mathcal{M}'_1$ to define $K_h : K_x \to K_y$ as the unique induced morphism with $m_y \circ K_h = C(h) \circ m_x$, $K_h \circ e_x = e_y \circ A(h)$ and $K_h \circ e'_x = e'_y \circ B(h)$.

$$
\begin{tikzcd}
& A(x) \arrow[dr, "e_x"] \arrow[drr, bend left, "f(x)"] \arrow[ddd, "A(h)"] & & \\
& & K_x \arrow[r, "m_x"] \arrow[ddd, "K_h"] & C(x) \arrow[ddd, "C(h)"] \\
B(x) \arrow[urr, "e'_x"] \arrow[urrr, bend right, "g(x)"'] \arrow[ddd, "B(h)"] & & & \\
& A(y) \arrow[dr, "e_y"] \arrow[drr, bend left, "f(y)"] & & \\
& & K_y \arrow[r, "m_y"] & C(y) \\
B(y) \arrow[urr, "e'_y"] \arrow[urrr, bend right, "g(y)"'] & & &
\end{tikzcd}
$$

Using the uniqueness property of the strong pair factorization in $\mathbf{C}$, we can show that K with $K(x) = K_x$, $K(h) = K_h$ is a functor and by construction e, e' and m are natural transformations. This means $(e, e') \in \mathcal{E}_1'^{funct}$ and $m \in \mathcal{M}_1'^{funct}$, i.e. this is an $\mathcal{E}_1'^{funct}$–$\mathcal{M}_1'^{funct}$ pair factorization of f and g.
The $\mathcal{E}_1'^{funct}$–$\mathcal{M}_1'^{funct}$ diagonal property can be shown as follows. Given $(e, e') \in \mathcal{E}_1'^{funct}$, $m \in \mathcal{M}_1'^{funct}$ and morphisms a, b, n in $[\mathbf{X}, \mathbf{C}]$, from the $\mathcal{E}_1'$–$\mathcal{M}_1'$ diagonal property in $\mathbf{C}$ we obtain a unique morphism $d_x : K(x) \to L(x)$ for $x \in \mathbf{X}$. It remains to show that $d = (d_x)_{x \in \mathbf{X}}$ is a natural transformation, i.e. we have to show for all $h : x \to y \in \mathbf{X}$ that $L(h) \circ d_x = d_y \circ K(h)$.

$$
\begin{tikzcd}
& & & & A(y) \arrow[ddrr, "e(y)"] \arrow[dddr, "a(y)"'] & & \\
A(x) \arrow[urrrr, bend left, "A(h)"] \arrow[drr, "e(x)"] \arrow[ddr, "a(x)"'] & & & & & & \\
& B(x) \arrow[r, "e'(x)"] \arrow[d, "b(x)"] \arrow[rrrr, bend left, "B(h)"] & K(x) \arrow[d, "n(x)"] \arrow[dl, dotted, "d_x"'] \arrow[rrrr, bend left, "K(h)"] & & & B(y) \arrow[r, "e'(y)"] \arrow[d, "b(y)"] & K(y) \arrow[d, "n(y)"] \arrow[dl, dotted, "d_y"'] \\
& L(x) \arrow[r, "m(x)"] \arrow[rrrr, bend right, "L(h)"'] & C(x) \arrow[rrrr, bend right, "C(h)"'] & & & L(y) \arrow[r, "m(y)"] & C(y)
\end{tikzcd}
$$

Consider the following diagram, where because of $(e(x), e'(x)) \in \mathcal{E}_1'$ and $m(y) \in \mathcal{M}_1'$ the $\mathcal{E}_1'$–$\mathcal{M}_1'$ diagonal property can be applied. This means there is a unique $k : K(x) \to L(y)$ with $k \circ e(x) = L(h) \circ a(x)$, $k \circ e'(x) = L(h) \circ b(x)$ and $m(y) \circ k = n(y) \circ K(h)$.

$$
\begin{tikzcd}
A(x) \arrow[drr, "e(x)"] \arrow[ddr, "L(h)\circ a(x)"'] & & \\
& B(x) \arrow[r, "e'(x)"] \arrow[d, "L(h)\circ b(x)"] & K(x) \arrow[d, "n(y)\circ K(h)"] \arrow[dl, dotted, "k"'] \\
& L(y) \arrow[r, "m(y)"] & C(y)
\end{tikzcd}
$$

For $L(h) \circ d_x$ we have: $L(h) \circ d_x \circ e(x) = L(h) \circ a(x)$, $L(h) \circ d_x \circ e'(x) = L(h) \circ b(x)$ and $m(y) \circ L(h) \circ d_x = C(h) \circ m(x) \circ d_x = C(h) \circ n(x) = n(y) \circ K(h)$. For $d_y \circ K(h)$ we have: $d_y \circ K(h) \circ e(x) = d_y \circ e(y) \circ A(h) = a(y) \circ A(h) = L(h) \circ a(x)$, $d_y \circ K(h) \circ e'(x) = d_y \circ e'(y) \circ B(h) = b(y) \circ B(h) = L(h) \circ b(x)$ and $m(y) \circ d_y \circ K(h) = n(y) \circ K(h)$. Thus, from the uniqueness of k it follows that $k = L(h) \circ d_y = d_y \circ K(h)$ and d is a natural transformation. □

Example 9

1. In **Graphs** × **Graphs**, the strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorization is constructed componentwise in **Graphs** (see item 5).
2. The category $\mathbf{Graphs_{TG}}$ inherits the strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorization from **Graphs**.
3. The category **PSets** $\tilde{=}$ $\{1\}\backslash$**Sets** inherits the strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorization from **Sets**, where $\mathcal{M}'$ is the class of all injective morphisms and $\mathcal{E}'$ is the class of jointly surjective morphisms.
4. The constructions in product and slice categories are examples for the construction of an $\mathcal{E}'$–$\mathcal{M}'$ pair factorization in comma categories; hence we can reuse the examples in items 1 and 2.
5. The category **Graphs** inherits the strong $\mathcal{E}'$–$\mathcal{M}'$ pair factorization from **Sets** in Example 8. It is constructed componentwise on the node and edge sets.

4 Conclusion and Future Work

The algebraic theory of graph transformations [5, 7] has been generalized recently to the framework of adhesive high-level replacement systems, which are based on adhesive [11, 12] and adhesive high-level replacement categories [3, 6]. It has been shown already that this kind of categories is closed under product, slice, coslice, comma and functor category constructions. The main contribution of this paper is to show under which conditions additional properties like initial pushouts, coproducts and pair factorizations, needed in the theory of adhesive high-level replacement systems, are preserved under these constructions. This avoids to check these properties explicitly for all the instantiations of adhesive high-level replacement categories.

The most important new results are the construction of initial pushouts in functor and comma categories, because in general initial pushouts cannot be constructed componentwise. Moreover pair factorizations, introduced in [3, 6] as modifications of $\mathcal{E}$–$\mathcal{M}$ factorizations, are in general not preserved by functor category constructions. For this reason we have introduced the new notion of a strong pair factorization, which requires in addition a diagonal property similar to that of $\mathcal{E}$–$\mathcal{M}$ factorizations. This allows to show that strong pair factorizations are preserved by all the category constructions including functor categories.

Concerning future work it remains to relax some of the conditions under which the additional properties are preserved. Moreover, it is interesting to analyze how to construct pushout complements in adhesive high-level replacement categories, where a characterization for the existence of pushout complements using initial pushouts is given already in [3].

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