Artificial Intelligence

Propositional Logic

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(slides based on Stuart Russell's AI course)

Motivation:

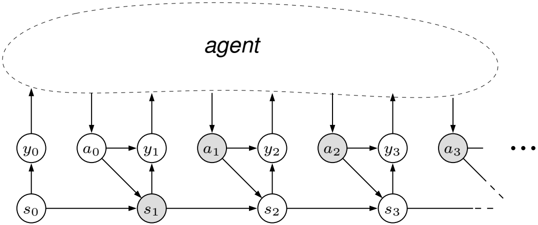
Most students will have learnt about propositional logic their first classes. It represents the simplest and most basic kind of logic. The main motivation to teach it really is as a precursor of first-order logic (FOL), which is covered in the next lecture. The intro of the next lecture motivates FOL in detail. The main point is that in recent years there were important developments that unified FOL methods with probabilistic reasoning and learning methods, which really allows to tackle novel problems.
In this lecture we go quickly over the syntax and semantics of propositional logic. Then we cover the basic methods for logic inference: fwd & bwd chaining, as well as resolution.

Syntax & Semantics

Outline

- Example: Knowledge-based agents & Wumpus world
- Logic in general—models and entailment
- Propositional (Boolean) logic
- Equivalence, validity, satisfiability
- Inference rules and theorem proving
  - forward chaining
  - backward chaining
  - resolution

Knowledge bases



- An agent maintains a knowledge base

  Knowledge base = set of sentences of a *formal* language

Wumpus World description

Performance measure

gold +1000, death -1000

-1 per step, -10 for using the arrow

Environment

Squares adjacent to wumpus are smelly

Squares adjacent to pit are breezy

Glitter iff gold is in the same square

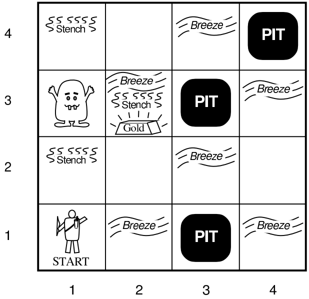
Shooting kills wumpus if you are facing it

The wumpus kills you if in the same square

Shooting uses up the only arrow

Grabbing picks up gold if in same square

Releasing drops the gold in same square

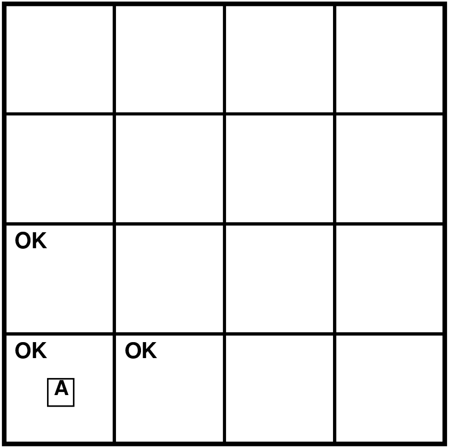


Actuators Left turn, Right turn,

Forward, Grab, Release, Shoot, Climb

Sensors Breeze, Glitter, Stench, Bump, Scream

Exploring a wumpus world



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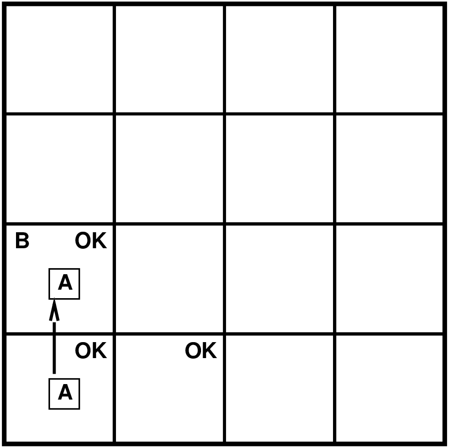

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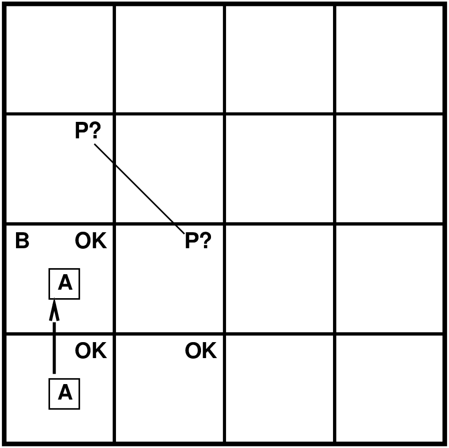

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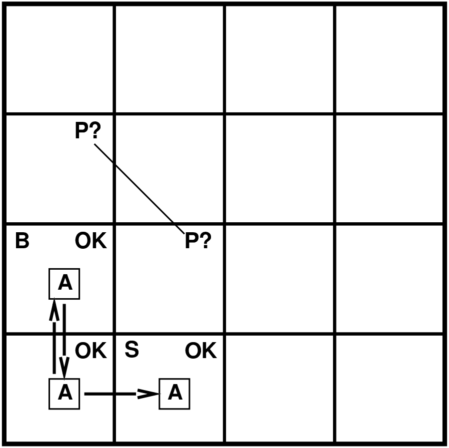

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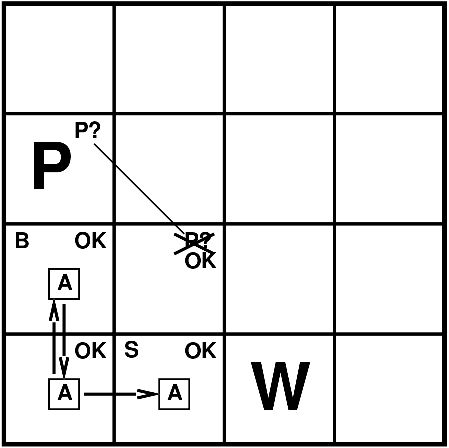

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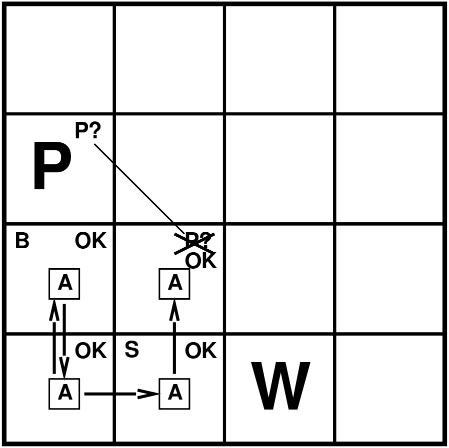

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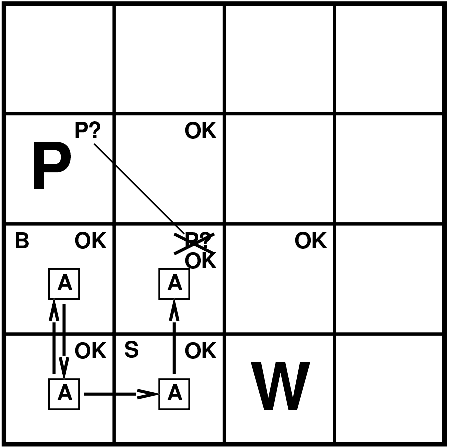

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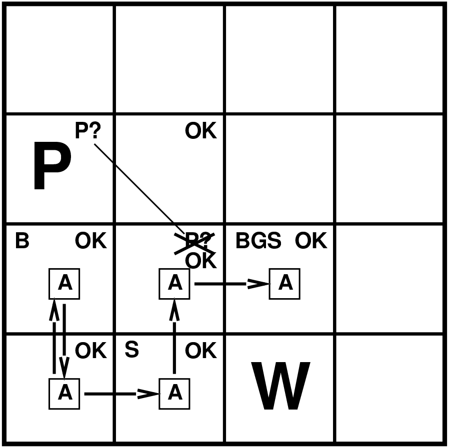
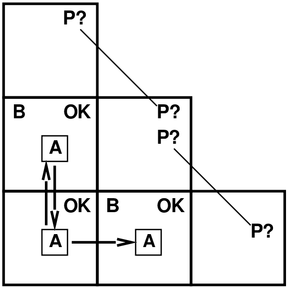


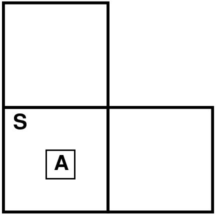
Other tight spots



Breeze in (1,2) and (2,1)

$\Rightarrow$ no safe actions

Assuming pits uniformly distributed, (2,2) has pit w/ prob 0.86, vs. 0.31



Smell in (1,1) $\Rightarrow$ cannot move

Can use a strategy of coercion:

shoot straight ahead

wumpus was there $\Rightarrow$ dead $\Rightarrow$ safe

wumpus wasn't there $\Rightarrow$ safe

Logic in general

- A Logic is a formal languages for representing information such that conclusions can be drawn
- The Syntax defines the sentences in the language
- The Semantics defines the "meaning" of sentences; i.e., define truth of a sentence in a world

  E.g., the language of arithmetic
  $x + 2 \geq y$ is a sentence; $x2 + y >$ is not a sentence
  $x + 2 \geq y$ is true iff the number $x + 2$ is no less than the number y
  $x + 2 \geq y$ is true in a world where $x = 7,\ y = 1$
  $x + 2 \geq y$ is false in a world where $x = 0,\ y = 6$

Notions in general logic

- A logic is a language, elements α are sentences
- A model m is a world/state description that allows us to evaluate $\alpha(m) \in \{\text{true}, \text{false}\}$ uniquely for any sentence α
  We define $M(\alpha) = \{m : \alpha(m) = \text{true}\}$ as the models for which α holds
- Entailment $\alpha \models \beta$: $M(\alpha) \subseteq M(\beta)$, "$\forall_m : \alpha(m) \Rightarrow \beta(m)$" (Folgerung)
- Equivalence $\alpha \equiv \beta$: iff ($\alpha \models \beta$ and $\beta \models \alpha$)
- A KB is a set (=conjunction) of sentences
- An inference procedure i can infer α from KB: $KB \vdash_i \alpha$
- soundness of i: $KB \vdash_i \alpha$ implies $KB \models \alpha$ (Korrektheit)
- completeness of i: $KB \models \alpha$ implies $KB \vdash_i \alpha$

Propositional logic: Syntax

$$
\begin{array}{lcl}
\langle \text{sentence} \rangle & \to & \langle \text{atomic sentence} \rangle \mid \langle \text{complex sentence} \rangle \\
\langle \text{atomic sentence} \rangle & \to & \text{true} \mid \text{false} \mid P \mid Q \mid R \mid \ldots \\
\langle \text{complex sentence} \rangle & \to & \neg \langle \text{sentence} \rangle \\
 & & \mid (\langle \text{sentence} \rangle \wedge \langle \text{sentence} \rangle) \\
 & & \mid (\langle \text{sentence} \rangle \vee \langle \text{sentence} \rangle) \\
 & & \mid (\langle \text{sentence} \rangle \ \Rightarrow \ \langle \text{sentence} \rangle) \\
 & & \mid (\langle \text{sentence} \rangle \ \Leftrightarrow \ \langle \text{sentence} \rangle)
\end{array}
$$

Propositional logic: Semantics

- Each model specifies true/false for each proposition symbol

  E.g. $P_{1,2}$ $P_{2,2}$ $P_{3,1}$

  true true false

  (With these symbols, 8 possible models, can be enumerated automatically.)

- Rules for evaluating truth with respect to a model m:

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true *and*	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true *or*	S_2	is true
$S_1 \Rightarrow S_2$	is true iff	S_1	is false *or*	S_2	is true
i.e.,	is false iff	S_1	is true *and*	S_2	is false
$S_1 \Leftrightarrow S_2$	is true iff	$S_1 \Rightarrow S_2$	is true *and*	$S_2 \Rightarrow S_1$	is true

- Simple recursive process evaluates an arbitrary sentence, e.g.,

  $\neg P_{1,2} \wedge (P_{2,2} \vee P_{3,1})$ = $\text{true} \wedge (\text{false} \vee \text{true}) = \text{true} \wedge \text{true} = \text{true}$

Notions in propositional logic – summary

- conjunction: $\alpha \wedge \beta$, disjunction: $\alpha \vee \beta$, negation: $\neg\alpha$
- implication: $\alpha \Rightarrow \beta \;\equiv\; \neg\alpha \vee \beta$
- biconditional: $\alpha \Leftrightarrow \beta \;\equiv\; (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$
  Note: $\models$ and $\equiv$ are statements about sentences in a logic; $\Rightarrow$ and $\Leftrightarrow$ are symbols in the grammar of propositional logic
- α valid: true for *any* model (allgemeingültig). E.g.,
  true; $A \vee \neg A$; $A \Rightarrow A$; $(A \wedge (A \Rightarrow B)) \Rightarrow B$
  Note: $KB \models \alpha$ iff $[(KB \Rightarrow \alpha)$ is valid$]$
- α unsatisfiable: true for *no* model. E.g., $A \wedge \neg A$;
  Note: $KB \models \alpha$ iff $[(KB \wedge \neg\alpha)$ is unsatisfiable$]$
- literal: A or $\neg A$, clause: disj. of literals, CNF: conj. of clauses
- Horn clause: symbol | (conjunction of symbols $\Rightarrow$ symbol), Horn form: conjunction of Horn clauses
  Modus Ponens rule: complete for Horn KBs $\frac{\alpha_1,\ldots,\alpha_n, \quad \alpha_1 \wedge \cdots \wedge \alpha_n \;\Rightarrow\; \beta}{\beta}$
  Resolution rule: complete for propositional logic in CNF, let "$\ell_i = \neg m_j$":

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

Logical equivalence

- Two sentences are logically equivalent iff true in same models:
  $\alpha \equiv \beta$ if and only if $\alpha \models \beta$ and $\beta \models \alpha$

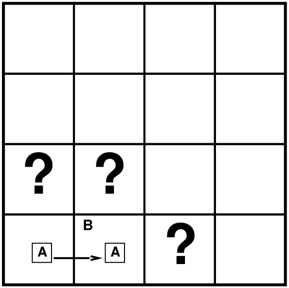
$$
\begin{aligned}
(\alpha \land \beta) &\equiv (\beta \land \alpha) \quad \text{commutativity of } \land \\
(\alpha \lor \beta) &\equiv (\beta \lor \alpha) \quad \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) &\equiv (\alpha \land (\beta \land \gamma)) \quad \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) &\equiv (\alpha \lor (\beta \lor \gamma)) \quad \text{associativity of } \lor \\
\neg(\neg\alpha) &\equiv \alpha \quad \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\beta \Rightarrow \neg\alpha) \quad \text{contraposition} \\
(\alpha \Rightarrow \beta) &\equiv (\neg\alpha \lor \beta) \quad \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) &\equiv ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) \quad \text{biconditional elimination} \\
\neg(\alpha \land \beta) &\equiv (\neg\alpha \lor \neg\beta) \quad \text{De Morgan} \\
\neg(\alpha \lor \beta) &\equiv (\neg\alpha \land \neg\beta) \quad \text{De Morgan} \\
(\alpha \land (\beta \lor \gamma)) &\equiv ((\alpha \land \beta) \lor (\alpha \land \gamma)) \quad \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) &\equiv ((\alpha \lor \beta) \land (\alpha \lor \gamma)) \quad \text{distributivity of } \lor \text{ over } \land
\end{aligned}
$$

Example: Entailment in the wumpus world

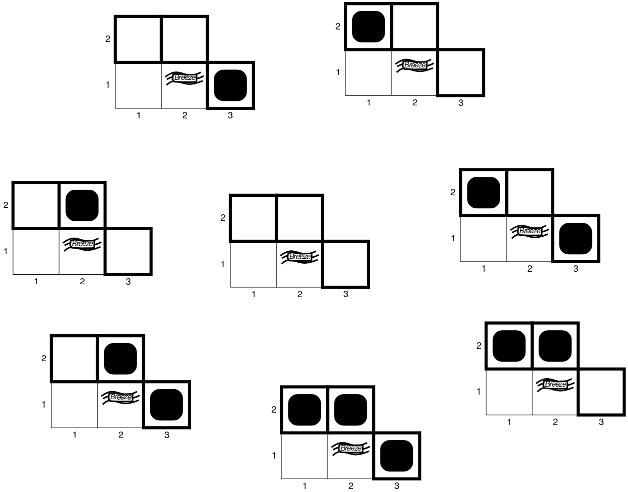
Situation after detecting nothing in [1,1], moving right, breeze in [2,1]

Consider possible models for ?s assuming only pits

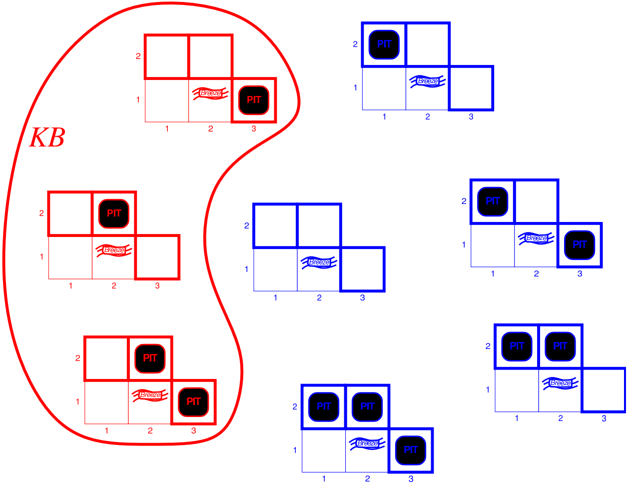
3 Boolean choices $\Rightarrow$ 8 possible models



Wumpus models

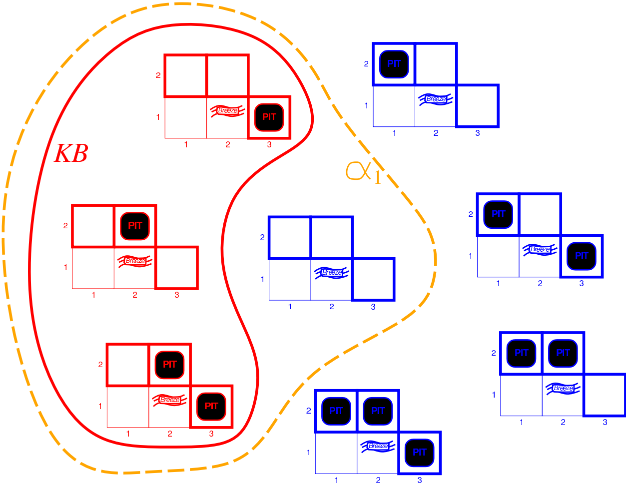


Wumpus models



KB = wumpus-world rules + observations

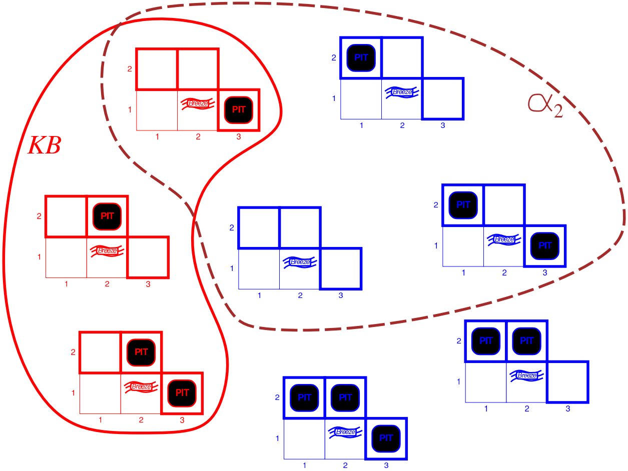
Wumpus models



KB = wumpus-world rules + observations

α_1 = "[1,2] is safe", $KB \models \alpha_1$, proved by model checking

Wumpus models



KB = wumpus-world rules + observations

α_2 = "[2,2] is safe", $KB \not\models \alpha_2$

Inference Methods

Inference

- Inference in the general sense means: Given some pieces of information (prior, observed variabes, knowledge base) what is the implication (the implied information, the posterior) on other things (non-observed variables, sentence)

- $KB \vdash_i \alpha$ = sentence α can be derived from KB by procedure i
  Consequences of KB are a haystack; α is a needle.
  Entailment = needle in haystack; inference = finding it

- Soundness: i is sound if
  whenever $KB \vdash_i \alpha$, it is also true that $KB \models \alpha$
  Completeness: i is complete if
  whenever $KB \models \alpha$, it is also true that $KB \vdash_i \alpha$

Preview: we will define a logic (first-order logic) which is expressive enough to say almost anything of interest, and for which there exists a sound and complete inference procedure. That is, the procedure will answer any question whose answer follows from what is known by the KB.

Inference by enumeration

$B_{1,1}$	$B_{2,1}$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	$P_{2,2}$	$P_{3,1}$	R_1	R_2	R_3	R_4	R_5	KB
false	*false*	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*false*	*false*
false	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*false*	*true*	*false*	*false*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
false	*true*	*false*	*false*	*false*	*false*	*false*	*true*	*true*	*false*	*true*	*true*	*false*
false	*true*	*false*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*false*	*true*	*false*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*false*	*true*	*true*	*true*	*true*	*true*	*true*	*true*	*true*
false	*true*	*false*	*false*	*true*	*false*	*false*	*true*	*false*	*false*	*true*	*true*	*false*
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
true	*true*	*true*	*true*	*true*	*true*	*true*	*false*	*true*	*true*	*false*	*true*	*false*

Enumerate rows (different assignments to symbols),
if KB is true in row, check that α is too

Inference by enumeration

Depth-first enumeration of all models is sound and complete

function TT-ENTAILS?(*KB*, α) **returns** *true* or *false*
  inputs: *KB*, the knowledge base, a sentence in propositional logic
    α, the query, a sentence in propositional logic

  symbols ← a list of the proposition symbols in *KB* and α
  return TT-CHECK-ALL(*KB*, α, *symbols*, [ ])

---

function TT-CHECK-ALL(*KB*, α, *symbols*, *model*) **returns** *true* or *false*
  if EMPTY?(*symbols*) **then**
    if PL-TRUE?(*KB*, *model*) **then return** PL-TRUE?(α, *model*)
    else return *true*
  else do
    $P \leftarrow$ FIRST(*symbols*); *rest* ← REST(*symbols*)
    return TT-CHECK-ALL(*KB*, α, *rest*, EXTEND(P, *true*, *model*)) **and**
      TT-CHECK-ALL(*KB*, α, *rest*, EXTEND(P, *false*, *model*))

$O(2^n)$ for n symbols

Proof methods

- Proof methods divide into (roughly) two kinds:

- Application of inference rules
  - Legitimate (sound) generation of new sentences from old
  - Proof = a sequence of inference rule applications
    Can use inference rules as operators in a standard search alg.
  - Typically require translation of sentences into a normal form

- Model checking
  truth table enumeration (always exponential in n)
  improved backtracking, e.g., Davis–Putnam–Logemann–Loveland (see book)
  heuristic search in model space (sound but incomplete)
  e.g., min-conflicts-like hill-climbing algorithms

Forward and backward chaining

- Applicable when KB is in Horn Form
- Horn Form (restricted)

  KB = *conjunction* of *Horn clauses*

  Horn clause =

  – proposition symbol; or

  – (conjunction of symbols) $\Rightarrow$ symbol

  E.g., $C \wedge (B \Rightarrow A) \wedge (C \wedge D \Rightarrow B)$

- Modus Ponens (for Horn Form): complete for Horn KBs

$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

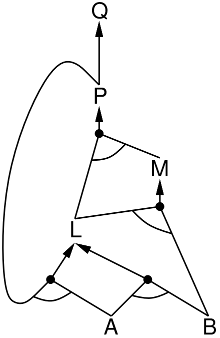
  Can be used with forward chaining or backward chaining.

- These algorithms are very natural and run in *linear* time

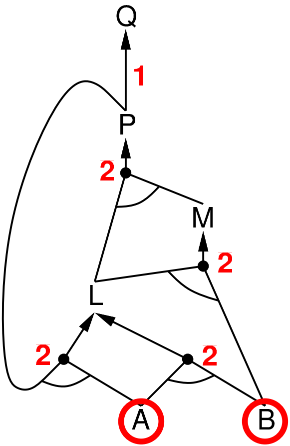
Forward chaining

- Represent a KB as a graph
- Fire any rule whose premises are satisfied in the KB,
  add its conclusion to the KB, until query is found

$$P \Rightarrow Q$$
$$L \wedge M \Rightarrow P$$
$$B \wedge L \Rightarrow M$$
$$A \wedge P \Rightarrow L$$
$$A \wedge B \Rightarrow L$$
$$A$$
$$B$$



Forward chaining example



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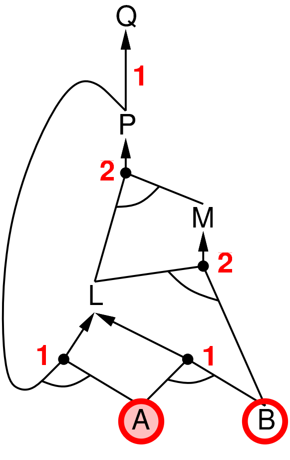

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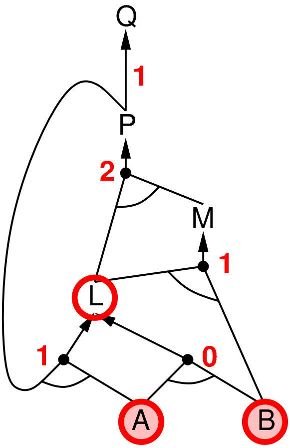

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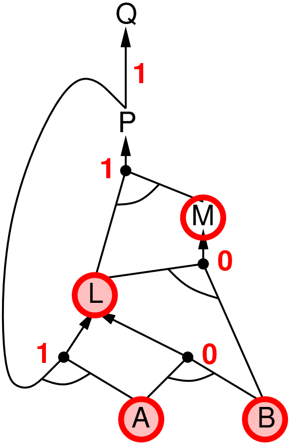

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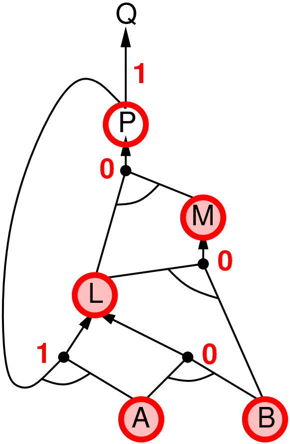

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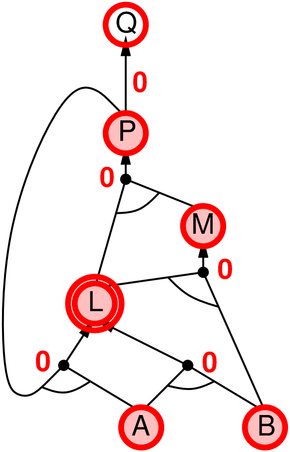

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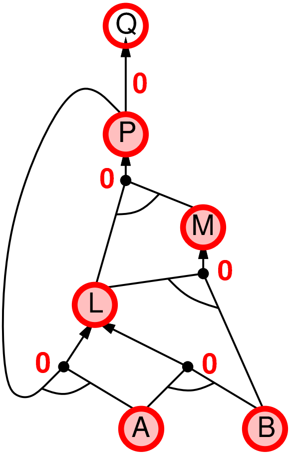

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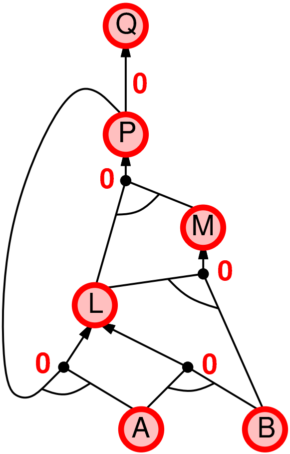


Forward chaining algorithm

```
function PL-FC-ENTAILS?(KB, q) returns true or false
   inputs: KB, the knowledge base, a set of propositional Horn clauses
           q, the query, a proposition symbol
   local variables: count, a table, indexed by clause, initially the number of premises
                    inferred, a table, indexed by symbol, each entry initially false
                    agenda, a list of symbols, initially the symbols known in KB

   while agenda is not empty do
       p ← POP(agenda)
       unless inferred[p] do
           inferred[p] ← true
           for each Horn clause c in whose premise p appears do
               decrement count[c]
               if count[c] = 0 then do
                   if HEAD[c] = q then return true
                   PUSH(HEAD[c], agenda)
   return false
```

Proof of completeness

FC derives every atomic sentence that is entailed by KB

1. FC reaches a fixed point where no new atomic sentences are derived
2. Consider the final state as a model m, assigning true/false to symbols
3. Every clause in the original KB is true in m

   Proof: Suppose a clause $a_1 \wedge \ldots \wedge a_k \Rightarrow b$ is false in m

   Then $a_1 \wedge \ldots \wedge a_k$ is true in m and b is false in m

   Therefore the algorithm has not reached a fixed point!
4. Hence m is a model of KB
5. If $KB \models q$, q is true in *every* model of KB, including m

General idea: construct any model of KB by sound inference, check α

Backward chaining

- Idea: work backwards from the query q:

  to prove q by BC,

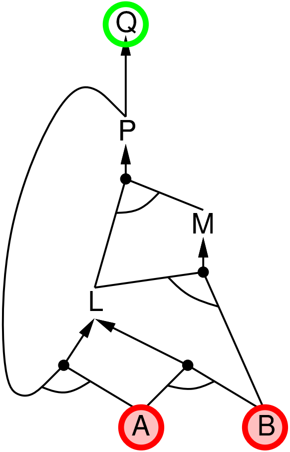
  check if q is known already, or

  prove by BC all premises of some rule concluding q
- Avoid loops: check if new subgoal is already on the goal stack
- Avoid repeated work: check if new subgoal

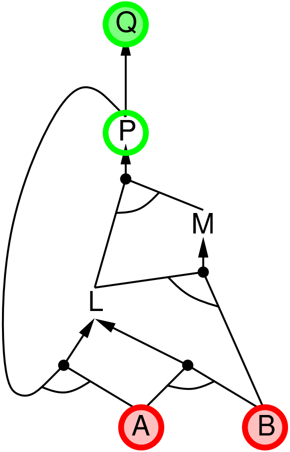
  1) has already been proved true, or

  2) has already failed

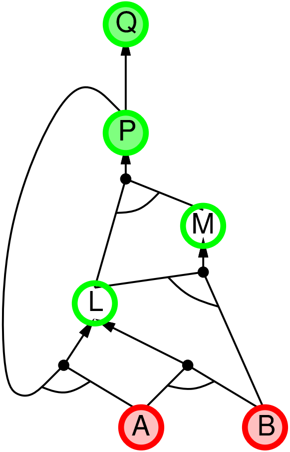
Backward chaining example



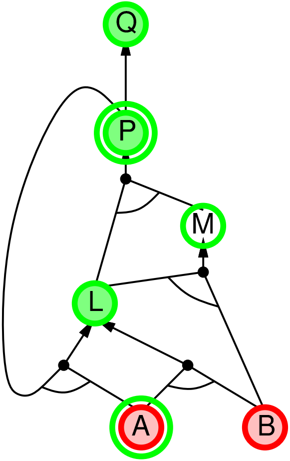
Backward chaining example



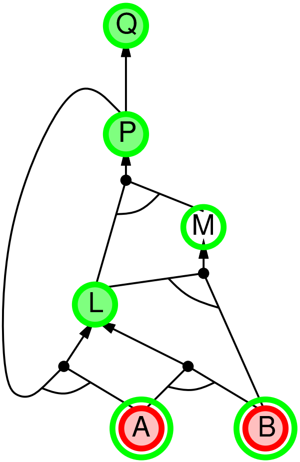
Backward chaining example



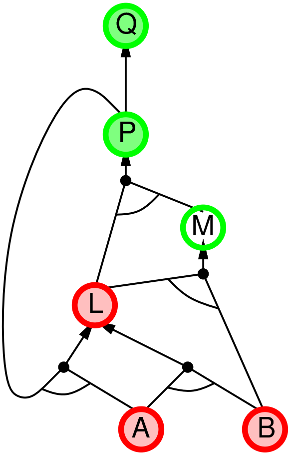
Backward chaining example



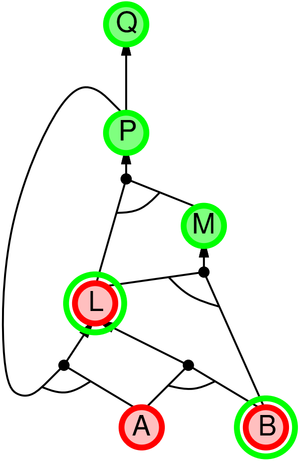
Backward chaining example



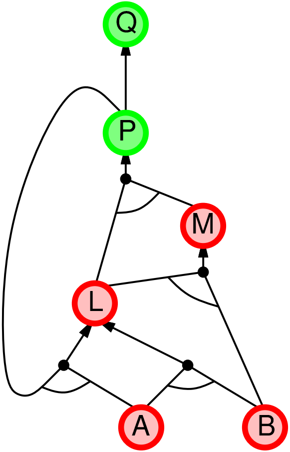
Backward chaining example



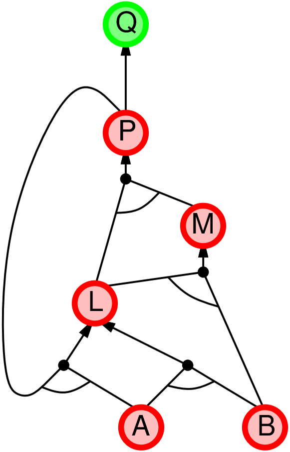
Backward chaining example



Backward chaining example



Backward chaining example



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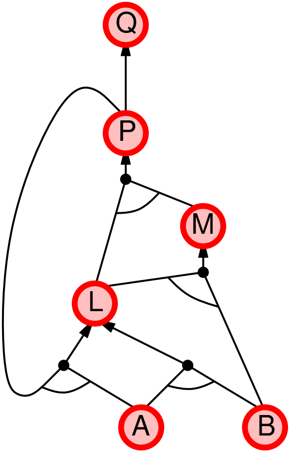


Forward vs. backward chaining

FC is data-driven, cf. automatic, unconscious processing,

e.g., object recognition, routine decisions

May do lots of work that is irrelevant to the goal

BC is goal-driven, appropriate for problem-solving,

e.g., Where are my keys? How do I get into a PhD program?

Complexity of BC can be *much less* than linear in size of KB

Resolution

- Conjunctive Normal Form (CNF—universal)

  conjunction of $\underbrace{\textit{disjunctions} \text{ of } \textit{literals}}_{\textit{clauses}}$

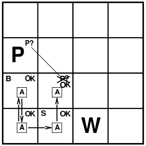
  E.g., $(A \vee \neg B) \wedge (B \vee \neg C \vee \neg D)$

- Resolution inference rule (for CNF): complete for propositional logic

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

  where ℓ_i and m_j are complementary literals.

- E.g., $$\frac{P_{1,3} \vee P_{2,2}, \qquad \neg P_{2,2}}{P_{1,3}}$$



- Resolution is sound and complete for propositional logic

Conversion to CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)$.

$$(B_{1,1} \Rightarrow (P_{1,2} \lor P_{2,1})) \land ((P_{1,2} \lor P_{2,1}) \Rightarrow B_{1,1})$$

2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \lor \beta$.

$$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg(P_{1,2} \lor P_{2,1}) \lor B_{1,1})$$

3. Move $\neg$ inwards using de Morgan's rules and double-negation:

$$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land ((\neg P_{1,2} \land \neg P_{2,1}) \lor B_{1,1})$$

4. Apply distributivity law ($\lor$ over $\land$) and flatten:

$$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})$$

Resolution algorithm

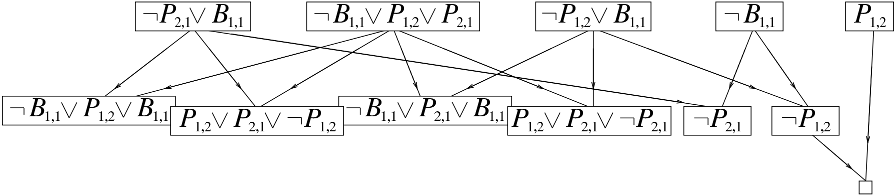
Proof by contradiction, i.e., show $KB \wedge \neg\alpha$ unsatisfiable

```
function PL-RESOLUTION(KB, α) returns true or false
   inputs: KB, the knowledge base, a sentence in propositional logic
           α, the query, a sentence in propositional logic

   clauses ← the set of clauses in the CNF representation of KB ∧ ¬α
   new ← { }
   loop do
      for each Ci, Cj in clauses do
         resolvents ← PL-RESOLVE(Ci, Cj)
         if resolvents contains the empty clause then return true
         new ← new ∪ resolvents
      if new ⊆ clauses then return false
      clauses ← clauses ∪ new
```

Resolution example

$$KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1} \qquad \alpha = \neg P_{1,2}$$



Summary

Logical agents apply inference to a knowledge base
to derive new information and make decisions

Basic concepts of logic:

- syntax: formal structure of sentences
- semantics: truth of sentences wrt models
- entailment: necessary truth of one sentence given another
- inference: deriving sentences from other sentences
- soundness: derivations produce only entailed sentences
- completeness: derivations can produce all entailed sentences

Wumpus world requires the ability to represent partial and negated information, reason by cases, etc.

Forward, backward chaining are linear-time, complete for Horn clauses

Resolution is complete for propositional logic

Propositional logic lacks expressive power