Optimization Algorithms
Weekly Exercises 5

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Winter 2024/25

1 Using the Lagrangian to solve a constrained problem analytically I

a) Consider the following constrained problem:

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^n x_i \quad \text{s.t.} \quad g(x) \le 0 \ , \quad g(x) = \begin{pmatrix} x^\top x - 1 \\ -x_1 \end{pmatrix} . \tag{1}$$

b) Evaluate all possible combinations of active/inactive constraints to find a point that fulfills the KKT conditions. Which is the optimum, and what are the optimal dual parameters? Note that $x \in \mathbb{R}^n$ (not 2D as previously).

c) Draw the feasible set and the optimal solution for $n = 2$.

2 Using the Lagrangian to solve a constrained problem analytically II

We consider again the following problem (which appeared in the very first exercise sheet):
In $\mathbb{R}^n$, a plane (through the origin) is described by the linear equation

$$c^\top x = 0 \ , \tag{2}$$

where $c \in \mathbb{R}^n$ parameterizes the plane and $x \in \mathbb{R}^n$ is a variable. Provide the matrix that describes the orthogonal projection of a point $x_0 \in \mathbb{R}^n$ onto this plane.

a) Formulate a constrained optimization problem that describes the projection. Solve this analytically, writing down the Lagrangian, and extract the projection matrix.

3 Lagrangian and dual function

(Taken roughly from 'Convex Optimization', Ex. 5.1)
Consider the optimization problem

$$\min x^2 + 1 \quad \text{s.t.} \quad (x-2)(x-4) \le 0 \tag{3}$$

with $x \in \mathbb{R}$.

a) Derive the optimal solution x^* and the optimal value $p^* = f(x^*)$ by hand. Write down the Lagrangian $L(x, \lambda)$.

b) For the same problem (3), plot $L(x, \lambda)$ over x for various values of $\lambda \ge 0$. The plots should verify the lower bound property $\min_x L(x, \lambda) \le p^*$, where p^* is the optimum value of the primal problem.

c) Derive the dual function $l(\lambda) = \min_x L(x, \lambda)$ and plot it (for $\lambda \ge 0$). Derive the dual optimal solution $\lambda^* = \text{argmax}_\lambda\, l(\lambda)$. Is $\max_\lambda l(\lambda) = p^*$ (strong duality)?

4 Phase I Optimization

Given an inequality-constrained mathematical program $\min_x f(x)$ s.t. $g(x) \le 0$, some solvers (like log barrier) require to be initialized with a *feasible* starting point x_0. However, finding a feasible initial point is not always easy. The term “Phase I Optimization” denotes the approach to formulate another optimization problem (which can easily be initialized feasible), so that solving this one leads to a feasible initialization of the original problem.

We will discuss standard approaches in the lecture. In this exercise, be creative yourself and come up with a “Phase I Optimization” formulation (which could be solved, e.g., by a log-barrier method).

Tip: If you lack ideas, check the term “slack variable” on Wikipedia. But please be creative also having other ideas.